M.Sc.(Mathematics) (NEP Pattern) Semester-I

NEP-64-3 / DSE-3 - Ordinary Differential Equations

P. Pages : 3 **GUG/W/23/15117**

Time : Three Hours *8044* Max. Marks : 80

Notes : 1. All questions are compulsory.
2. Each questions carry equal marks.

UNIT – I

1. a) Let ϕ be any solution of $L(y) = y'' + a_1 y' + a_2 y = 0$ on an interval I containing a point x_0. Then prove that for all x in I **8**

$$\|\phi(x_0)\| e^{-k|x-x_0|} \leqq \|p(x)\| \leqq \|\phi(x_0)\| e^{k|x-x_0|}$$

Where

$$\|\phi(x)\| = \left[|\phi(x)|^2 + |\phi'(x)|^2 \right]^{1/2}, k = 1 + |a_1| + |a_2|.$$

b) Prove that ϕ_1, ϕ_2 of L(y) = 0 are linearly independent on an interval I if, and only if, $W(\phi_1, \phi_2)(x) \neq 0$ for all x in I. **8**

OR

c) If ϕ_1, ϕ_2 are two solutions of $L(y)' = y'' + a_1 y' + a_2 y = 0$ on an interval I containing a point x_0, then prove that $w(\phi_1, \phi_2)(x) = e^{-a_1(x-x_0) w(\phi_1,\phi_2)(x_0)}$ **8**

d) Let ϕ be any solution of $L(y) = y^{(n)} + a_1 y^{(n-1)} + ... + a_n y = 0$ on an interval I containing a point x_0. Then Prove that for all x in I, $\|\phi(x_0)\| e^{-k|x-x_0|} \leqq \|\phi(x)\| \leqq \|\phi(x_0)\| e^{k|x-x_0|}$, Where $k = 1 + |a_1| + ... + |a_n|$ **8**

UNIT – II

2. a) Let $\phi_1, ..., \phi_n$ be the n solutions L(y) = 0 satisfying $\phi_i^{(i-1)}(x_0) = 1, \phi j^{(j-1)}(x_0) = 0,\ j \neq i$. If ϕ is any solution of L(y) = 0 on I, then prove that there are n constant $c_1, c_2, ..., c_n$ such that $\phi = c_1 \phi_1 + ... + c_n \phi_n$. **8**

b) Prove that if $\phi_1, ... \phi_n$ are n solutions of L(y) = 0 on an interval I, they are linearly independent there if and only if, $w(\phi_1, ..., \phi_n)(x) \neq 0$ for all x in I. **8**

OR

c) Show that the coefficient of x^n in $p_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$ is $\frac{(2n)!}{2^n (ni)^2}$ **8**

d) Solve the Besels equation of order α, where α is a constant and $\text{Re}\ \alpha \geq 0$. **8**

UNIT – III

3. a) Prove that a function ϕ is a solution of the initial value problem $y' = f(x, y), y(x_0) = y_0$ on an interval I if and only if it is a solution of the integral equation $y = y_0 + \int_{x_0}^{x} f(t, y)\,dt$ on I. **8**

b) Let M, N be two real-valued functions which have continuous first partial derivatives on some rectangle $R : |x - x_0| \leqq a,\ |y - y_0| \leqq b$. The prove that the equation $M(x, y) + N(x, y)y' = 0$ is exact in R if, and only if, $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. **8**

OR

c) Consider the initial value problem $y' = 3y + 1, y(0) = 2$. Show that all the successive approximations $\phi_0, \phi_1, ...$ exists for all real x. **8**

d) Suppose F is a real-valued continuous function on the plane $|x| < \infty, |y| < \infty$, which satisfies a Lipschitz condition on each strip $S_a : |x| \leqq a$, $|y| < \infty$, where a is any positive number. Then prove that every initial value problem $y' = f(x, y), y(x_0) = y_0$ has a solution which exists for all x. **8**

UNIT – IV

4. a) Write a note on the following some special equations. **8**

i) The equation $y'' = f(x, y')$

ii) The equation $y'' = f(y, y'')$

b) Suppose f is a vector – valued function defined for (x,y) on a set S of the form $|x - x_0| \leqq a, | y - y_0 | \leqq b, (a, b > 0)$ or of the form $|x - x_0| \leqq a, |y| < \infty, (a > 0)$. If $\partial f / \partial y_k\ (k = 1, ..., n)$ exists, is continuous on S, and there is a constant $k > 0$ such that $\left|\frac{\partial f}{\partial y_k}(x, y)\right| \leqq k, (k = 1, ..., n)$ for all (x,y) in S, then prove that f satisfies a Lipschitz condition on s with Lipschitz constant k. **8**

OR

c) Let f be a complex – valued continuous function defined on $R : |x - x_0| \leqq a, |y - y_0| \leq b, (a, b \geq 0)$ such that $|f(x, y)| \leqq N$ for all (x,y) in R. Suppose there exists a constant L > 0 such that $|f(x, y) - f(x, z)| \leqq L|y - z|$ for all (x,y) and (x,z) in R. Then prove that there exists one, and only one, solution ϕ of $y^{(n)} = f\left(x, y, y', ..., y^{(n-1)}\right)$ on the interval $I = |x - x_0| \leqq \min|a, b / m|, \left(M = N + b + |y_0|\right)$ Which satisfies $\phi(x_0) = a_1, \phi'(x_0) = a_2, ..., \phi^{(n-1)}(x_0) = an\ \left(y_0 = (a_1, ..., a_n)\right)$ **8**

d) Let F be a continuous vector-valued function defined on $R: |x - x_0| \leqq a, \ |y - y_0| \leqq b$ (a, b >0) and suppose F satisfies a Lipschitz condition on R. If M is a constant such that $|f(x, y)| \leqq M$ for all (x,y) in R. Then prove that the successive approximations $\{\phi_k\}, (k = 0, 1, 2...)$ given by $\phi_0(x) = y_0$ converges on the interval $I: |x - x_0| \leq \alpha$ = minimum $\{a, b/m\}$, to a solution ϕ of the initial value problem $y' = f(x, y) + y(x_0) = y_0$ on I. **8**

5. Solve the following,

a) Verify that the solutions $\phi(x) = e^{-\sin x}$ for the differential equations $y' + (\cos x) y = 0$. **4**

b) Consider the equation $y'' + \frac{1}{x} y' - \frac{1}{x^2} y = 0$. Find two linearly independent solutions for x > 0, and prove that they are linearly independent: **4**

c) Find all real-valued solution of the **4**

$$y' = \frac{x + x^2}{y - y^2}$$

d) Solve the differential equation $y^2 y'' = y'$. **4**

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