M.Sc. - I (Mathematics) (NEP Pattern) Semester-I

NEP-64-2 / DSE-2 - Paper-I : Real Analysis

P. Pages : 2 **GUG/W/23/15116**

Time : Three Hours * 8 0 4 3 * Max. Marks : 80

Notes : 1. Solve all **five** questions.
2. All questions carry equal marks.

UNIT – I

1. a) Let $\{f_n\}_{n=1}^{n=\infty}$ be a sequence of real value function on a set E. Then prove that $\{f_n\}_{n=1}^{n=\infty}$ is converges uniformly on E iff for every $\in > 0$ there exist an integer N such that $m \ge N,\ n \ge N\ \ x \in E \Rightarrow |f_m(x) - f_n(x)| \le \in$ **8**

b) If $f_n'(x)$ exist for each $x \in [a, b]$ and such that $f_n(x_0)$ converges for some point $x_0 \in [a, b]$. Then prove that **8**

i) $\{f_n\}$ converges uniformly on [a, b] through a function f.

ii) $f_n'(x) = \lim\limits_{n \to \infty} f_n'(x),\ a \le x \le b$

OR

c) If $\sum_{n=0}^{\infty} a_n = L$ and if $\lim\limits_{n \to \infty} n\, a_n = 0$. Then prove that $\sum_{n=0}^{\infty} a_n$ converges L. **8**

d) Prove that there exist a real continuous function on the real line which is nowhere differentiable. **8**

UNIT – II

2. a) Let A be an open in R^m. Suppose that the partial derivative $D_j f_i(x)$ of the component function of f exist at each point x of A and are continuous on A. Then prove that f is differentiable at each point of A. **8**

b) State and prove the inverse function theorem. **8**

OR

c) Let $A \subset R^m$ and $B \subset R^n$, Let $f : A \to R^n$ and $g : B \to R^p$, with $f(A) \subset B$. Suppose f(a) = b., If f is differentiable a and g is differentiable at b. Then prove that composite function $g \circ f$ is differentiable at a, furthermore, $D(g \circ f)(a) = Dg(b) \cdot Df(a)$. Where the indicated product is matrix multiplication. **8**

d) State and prove implicit function theorem. **8**

UNIT – III

3. a) Let Q be a rectangle ; let $f : Q \to R$ be a bounded function. Then prove that $\underline{\int}_Q f \le \overline{\int}_Q f$; equality holds if and only if given by $\in > 0$, there exist a corresponding partition P of Q for which $U(f, P) - L(f, P) < \in$. **8**

b) Let Q be a rectangle in R^n ; let $f: Q \to R$ be a bounded function. Let D be the set of points of Q at which f fails to be continuous. Then prove that $\int_Q f$ exist if and only if D has measure zero in R^n. **8**

OR

c) State and prove Fubini's theorem. **8**

d) Let A be open in R^n, let $f: A \to R$ be continuous. Choose a sequence C_N of compact rectifiable subset of A whose union is A such that $C_N \subset C_{N+1}$ for each N. Then prove that f is integrable over A if and only if sequence $\int_{C_N} |f|$ is bounded. In this $\int_A f = \lim_{N\to\infty} \int_{C_N} f$. **8**

UNIT – IV

4. a) Let $I = [a, b]; g: I \to R$ be a function of class C^1 with $g'(x) \neq 0$ for $x \in (a, b)$. Then the set g(I) is a closed set with interval J with end points g(a) and g(b). If $f: J \to R$ is continuous, then prove that $\int_{g(a)}^{g(b)} f = \int_a^b (f \cdot g) g'$ **8**

b) State and prove change of variable theorem. **8**

OR

c) Let A be open in R^n ; let $f: A \to R$ be continuous. Let (ϕ_i) be a partition of unity on A having compact support. Then prove that integral $\int_A f$ exists if and only if the series $\sum_{i=1}^{\infty} \left[\int_A \phi_i |f|\right]$ converges. **8**

d) Let $h: R^n \to R^n$ be a map such that h(0) = 0. Prove that **8**

i) The map h is an isometry if and only if preserves dot products.

ii) The map h is an isometry if and only it is orthogonal transformation.

5. a) Prove that able summability is regular. **4**

b) Let $A \subset R^m$; Let $f: A \to R^n$, if f is differentiable at a, then prove that f is continuous at a. **4**

c) Let I = [0, 1], let $f: I \to R$ be defined by **4**

$$\begin{cases} f(x) = 0 \text{ if } x \text{ is rational} \\ f(x) = 1 \text{ if } x \text{ is irrational} \end{cases}$$

Show that f is not integrable over I.

d) Write the statement of Existence of a partition of unity. **4**

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