M.Sc.- I (Mathematics) (NEP Pattern) Semester-I

NEP-63 / DSC-3 - Major - Paper-III : Linear Algebra

P. Pages : 3 **GUG/W/23/15114**

Time : Three Hours * 8 0 4 1 * Max. Marks : 80

Notes : 1. Solve all **five** questions.
2. All questions carry equal marks.

UNIT – I

1. a) Let V and V' be vector spaces over K, and let $T : V \to V'$ be a linear transformation. Then prove that **8**
i) $T(0) = 0$
ii) $T(-v) = -T(v),\ v \in V,$
iii) $T(U)$ is a subspace of V', whenever U is a subspace of V.
iv) $T^{-1}(U')$ is a subspace of V, whenever U' is a subspace of V'.

b) Let V and V' be vector spaces over K, and let $T \in L(V, V')$. Prove that: **8**
i) if W and W' are subspaces of V and v' respectively, and $T(W) \subseteq W'$, then T induces a linear transformation
$\bar{T} : V / W \to V' / W'$ defined by $\bar{T}(v + W) = T(v) + W'$
ii) If T is subjective, then $v / \ker T \simeq V'$.

OR

c) Let $V_1, ..., V_m$ be vector spaces over a field K. Then prove that $V = V_1 \oplus .... \oplus V_m$ is finite dimensional if and only if each V_i is finite dimensional. In this case **8**
$\dim V_1 \oplus .... \oplus V_m = \dim V_1 + ... + \dim V_m$

d) Prove that for two finite dimensional vector space V and V' over K, $V \simeq V'$ if and only if $\dim V = \dim V'$. **8**

UNIT – II

2. a) Let V be a finite dimensional vector space over K of dimension n and let T be a linear operator on V. If $m_T(x) = p(x)^r$, where p(x) is a monic irreducible polynomial of degree m, then prove that m divides n. **8**

b) Obtain the minimal polynomial for the matrix **8**

$$\begin{bmatrix} 1 & 1 & -2 & -2 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 1 & 0 \\ -1 & -2 & 0 & 1 \end{bmatrix}$$

OR

c) Prove that the eigenvectors corresponding to distinct eigenvalues of a linear operator are linearly independent. **8**

d) Prove that two diagonalizable operators S and T on V are simultaneously diagonalizable if and only if they commute. **8**

UNIT – III

3. a) Let V be an inner product space over f and let $u, \nu \in V$. Prove that **8**

i) $\| u \pm \nu \|^2 = \| u \|^2 \pm 2\,\mathrm{Re}(u, \nu) + \| \nu \|^2$ where Rez denotes the real part of the complex number z.

ii) $\| u + \nu \|^2 + \| u - \nu \|^2 = 2\,\| u \|^2 + 2\,\| \nu \|^2$

iii) $\| \lambda u \| = | \lambda |\, \| u \|$ for all $\lambda \in f$.

iv) $|(u, \nu)| \le \| u \|\, \| \nu \|$

b) Let $\{x_1, x_2, ..., x_n\}$ be a sequence of linearly independent vectors in an inner product space V. Then prove that there is a sequence of orthonormal vectors $\{y_1, y_2, ...\}$ such that for every n $\langle x_1, x_2, ..., x_n \rangle = \langle y_1, ..., y_n \rangle$ **8**

OR

c) Let V and W be finite dimensional inner product spaces and let $T \in L(V, W)$. Then prove that there exists a unique linear mapping $T^* = W \to V$ such that for all $\nu \in V$ and $w \in W$ $(T\nu, w) = (\nu, T^* w)$ **8**

d) Let T be a unitary operator on V, dim V = n. If B_1 and B_2 are ordered orthonormal bases of V, then prove that ${}_{B_2}[T]_{B_1}$ is a unitary matrix. **8**

UNIT – IV

4. a) Prove that if a bilinear form is reflexive then it is either symmetric or alternating. **8**

b) Let A and B be invertible matrices of the same size. Then prove that the following are equivalent: **8**

i) A and B are congruent

ii) A^{-1} and B^{-1} are congruent

iii) A^+ and B^+ are congruent.

OR

c) Prove that a symmetric bilinear form on a finite dimensional vector space over a field K of characteristic not equal to 2 is diagonalizable. **8**

d) Let ϕ be a reflexive bilinear form on a vector space V over K. If S is a finite dimensional anisotropic subspace of V, then prove that $V = S \oplus S^1$. **8**

5. a) Let W_1 and W_2 be subspaces of a vector space V. Then prove that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$. **4**

b) Let V be a finite dimensional vector space over K and let T be a linear operator on V, then prove that a scalar λ in K is an eigen value of t if and only if $\det(T - \lambda I) = 0$. **4**

c) Let W be a subspace of a finite dimensional inner product space V. Then prove that $V = W \oplus W^{\perp}$. **4**

d) Define: **4**

i) Bilinear form

ii) Bilinear space

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