M.Sc. First Year (Mathematics) (NEP Pattern) Semester-I

NEP-61 / DSC-1 - Paper-I - Advanced Abstract Algebra

P. Pages : 2 **GUG/W/23/15112**

Time : Three Hours * 8 0 3 9 * Max. Marks : 80

Notes : 1. Solve all **five** questions.
2. Each question carry equal marks.

UNIT – I

1. a) Let H and K be normal subgroups of G and $K \subset H$. Then prove that $(G/K)/(H/K) \cong G/H$. **8**

b) Let G be a group, and let G' be the derived group of G. Then prove that **8**
i) G' is a normal subgroup of G.
ii) G/G' is abelian.
iii) If H is a normal subgroup of G, then G/H is abelian if and only if $G' \subset H$.

OR

c) Let $\phi : G \to G'$ be a homomorphism of groups. Then prove that $G / \text{Ker}\ \phi \cong \text{Im}\ \phi$. **8**

d) Let G be a finite group acting on a finite set X. Then prove that the number k of orbits in X under G is $k = \frac{1}{|G|} \sum_{g \in G} |X_g|$. **8**

UNIT – II

2. a) Prove that any two composition series of a finite group are equivalent. **8**

b) Prove that a group G is nilpotent if and only if G has a normal series $\{e\} = G_0 \subset G_1 \subset \cdots \subset G_m = G$ such that $G_i / G_{i-1} \subset Z(G / G_{i-1})$ for all i = 1,…,m. **8**

OR

c) Prove that the alternating group A_n is generated by the set of all 3-cycles in S_n. **8**

d) If a permutation $\sigma \in S_n$ is a product of r transpositions and also a product of s transpositions, then prove that r and s are either both even or both odd. **8**

UNIT – III

3. a) Let G be a finite group, and let p be a prime. If p^m divides |G|, then prove that G has a subgroup of order p^m. **8**

b) Let A be a finite abelian group, and let p be prime. If p divides |A|, then prove that A has an element of order P. **8**

OR

c) Let G be a group of order pq, where p and q are prime numbers such that p > q and q does not divide (p – 1). Then prove that G is cyclic. **8**

d) Prove that there are no simple groups of orders 63,56, and 36. **8**

UNIT – IV

4. a) Let f be a homomorphism of a ring R into a ring S with kernel N. Then prove that $R / N \cong \operatorname{Im} f$. **8**

b) Prove that in a nonzero commutative ring with unity, an ideal M is maximal if and only if R/M is a field. **8**

OR

c) If a ring R has unity, then prove that every ideal I in the matrix ring R_n is of the form A_n, where A is some ideal in R. **8**

d) If R is a commutative ring, then prove that an ideal P in R is prime if and only if $ab \in P,\ a \in R,\ b \in R$ implies $a \in P$ or $b \in P$. **8**

5. a) Define Normal subgroup and normalizer of a nonempty subset S of a group. **4**

b) Define solvable group and nilpotent group. **4**

c) State first, second and third Sylow theorems. **4**

d) Define : Prime ideal and maximal ideal. **4**

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