M.Sc.-II (Mathematics) (New CBCS Pattern) Semester - IV

PSCMTH16 - Dynamical Systems

P. Pages : 2 **GUG/S/23/13767**

Time : Three Hours * 2 5 0 3 * Max. Marks : 100

Notes : 1. Solve all **five** questions.
2. All questions carry equal marks.

UNIT – I

1. a) Let the function $f: W \to E$ be C^1. Then prove that f is locally Lipschitz. **10**

b) Let a C^1 map $f: W \to E$ be given. Suppose two solutions u(t), v(t) of $x^1 = f(x)$ are defined on the same open interval J containing to and satisfy $u(t_0) = v(t_0)$ then prove that $u(t) = v(t)$ for all $t \in J$. **10**

OR

c) Let $W \subset E$ be open, Let $f: W \to E$ be a C^1 map. Let y(t) be a solution on a maximal open interval $J = (\alpha, \beta) \subset R$ with $\beta < \infty$. Then prove that given any compact set $K \subset W$, there is some $t \in (\alpha, \beta)$ with $y(t) \notin K$. **10**

d) Prove that Ω is an open set in $R \times W$ and $\phi : \Omega \to W$ is a continuous map. **10**

UNIT – II

2. a) Discuss the motion of pendulum moving in a vertical plane as an example of non-linear sink. **10**

b) Let $\bar{x}$ be an isolated minimum of V. Then prove that $\bar{x}$ is an asymptotically stable equilibrium of the gradient system $x' = -\text{grad} V(x)$ **10**

OR

c) There exists $\delta > 0$ such that if U is the closed ball $B_\delta(0) \subset W,$ then prove that for all $z = (x, y) \in C \cap U,$ **10**

a) $\langle x, f_1(x,y) \rangle - \langle y, f_2(x,y) \rangle > 0$ if $x \neq 0$ and

b) There exists $\alpha > 0$ with $\langle f(z), z \rangle \geq \alpha |z|^2$

d) Prove that Let $V: W \to R$ be a C^2 function (that is, DV : $W \to E^*$ is C^1, or V has continuous second partial derivatives) on an open set W in a vector space E with an inner product. **10**

i) $\bar{x}$ is an equilibrium point of the differential equation $x' = -\text{grad } V(x)$ iff $DV(\bar{x}) = 0$

ii) If $x(t)$ is a solution of $x' = -\text{grad } V(x)$, then $\frac{d}{dt} V(x(t)) = -\left|\text{grad } V(x(t))\right|^2$

iii) If x(t) is not constant, then V(x(t)) is a decreasing function of t.

UNIT – III

3. a) Prove that **10**

i) If x and z are on the same trajectory, then $L_w(x) = Lw(z)$, similarly for α - limits.

ii) If D is a closed positively invariant set & $Z \in D$, then $L_w(z) \subset D$, similarly for negatively invariant sets & $\alpha -$ limits.

b) Prove that let S be a local section at 0 & suppose $\phi_{to}(z_0) = 0$. There is an open set $U \subset W$ containing z_0 and a unique C^1 map $\tau : U \to R$ such that $\tau(z_0) = to$ and $\phi_{\tau(x)}(x) \in S$ for all $x \in U$ **10**

OR

c) Prove that a non empty compact limit set of a C^1 planar dynamical system, which contains no equilibrium point, is a closed orbit. **10**

d) Prove that every trajectory of the Volterra - Lotka equations $x' = (A - B_y)x,\ y' = (cx - D)y,$ A, B, C, D > 0 is a closed orbit (except the equilibrium Z and the coordinate axes). **10**

UNIT – IV

4. a) Let γ be an asymptotically stable closed orbit of period λ. Then prove that γ has a neighborhood $U \subset W$ such that every point of U has asymptotic period λ. **10**

b) Let $g : S_0 \to S$ be a Poincare map for γ, Let $x \in S_0$ be such that $\lim_{n\to\infty} g^n(x) = 0$. Then prove that $\lim_{n\to\infty} d(\varphi_t(x), \gamma) = 0$ **10**

OR

c) Let $A : J \to L(E)$ be a continuous map from an open interval J to the space of linear operators on E. Let $(t_0, u_0) \in J \times E$. Then prove that the initial value problem $x' = A(t)x,\ x(t_0) = u_0$ has a unique solution on all of J. **10**

d) Let $O \in E$ be a sink for a C^1 vector field $f : W \to E$ where W is an open set containing O. There exists an inner product on E, a number r > 0, and a neighborhood $\eta \subset \nu(w)$ of f such that the following holds: for each $g \in \eta$ there is a sink $a = a(g)$ for g such that the set $B_r = \{x \in E \,|\, |x| \le r\}$ contains a, is in the basin of a, and is positively invariant under the flow of g. **10**

5. a) Explain dynamical system with example. **5**

b) Define **5**

i) Stable equilibrium ii) Asymptotically stable.

c) Explain growth rate of the population at time t. **5**

d) Define structural stability. **5**

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