M.Sc. - II (Mathematics) New CBCS Pattern Semester-III

PSCMTH14D - Commutative Algebra (Optional)

P. Pages : 2 **GUG/W/23/13761**

Time : Three Hours Max. Marks : 100

Notes :
1. Solve all **five** questions.
2. Each question carry equal marks.

UNIT – I

1. a) Prove that every non-zero ring *A* has at least one maximal ideal. **10**

b) Prove that the set $\mathfrak{R}$ of all nilpotent elements in a ring *A* is an ideal, and $A / \mathfrak{R}$ has no nilpotent element $\neq 0$. **10**

OR

c) Let *A* be a nonzero ring. Then prove that the following are equivalent: **10**
i) *A* is a field:
ii) The only ideals in *A* are 0 and (1):
iii) Every homomorphism of *A* into a non-zero ring *B* is injective.

d) **10**
i) Let $p_1, p_2 ...., p_n$ be prime ideals and let a be an ideal contained in $\bigcup_{i=1}^{n} p_i$. Then prove that $a \subseteq p_i$ for some i.

ii) Let $a_1, a_2, ..., a_n$ be ideals and p be a prime ideal containing $\bigcap_{i=1}^{n} a_i$. Then prove that $p \supseteq a_i$ for some i. If $p = \bigcap_{i=1}^{n} a_i$, then prove that $p = a_i$ for some i.

UNIT – II

2. a) Let M, N, P be A – modules. Then prove that there exists unique isomorphism $(M \oplus N) \otimes P \to (M \otimes P) \oplus (N \otimes P)$ **10**

b) **10**
i) Let $M' \xrightarrow{u} M \xrightarrow{v} M'' \to 0$ be a sequence of A-modules and homomorphisms. Then prove that this sequence is exact if and only if for all A-modules N, then sequence. $0 \to Hom(M'', N) \xrightarrow{\bar{v}} Hom(M, N) \xrightarrow{\bar{u}} Hom(M', N)$ is exact.

ii) Let $0 \to N' \xrightarrow{u} N \xrightarrow{v} N''$ be a sequence of A-modules and homomorphisms. Then prove that this sequence is exact if and only if for all A-modules M, the sequence. $0 \to Hom(M, N') \xrightarrow{\bar{u}} Hom(M, N) \xrightarrow{\bar{v}} Hom(M, N'')$ is exact. **10**

OR

c) **10**
i) If $L \supseteq M \supseteq N$ are A-modules. Then prove that $(L / N) / (M / N) \cong L / M$
ii) If M_1, M_2 are submodules of M, then prove that $(M_1 + M_2) / M_1 \cong M_2 / (M_1 \cap M_2)$.

d) Let M, N be A-modules. Then prove that there exists a pair (T, g) consisting of an A-module T and an A-bilinear mapping $g: M \times N \to T$, with the following property: Given any A-module P and any A-bilinear mapping $f: M \times N \to P$, there exists a unique A-linear mapping $f': T \to P$ such that $f = f' \cdot g$. **10**

Moreover, if (T, g) and (T', g') are two pairs with this property, then there exists a unique isomorphism $j: T \to T'$ such that $j \cdot g = g'$.

UNIT – III

3. a) For any A-module M, prove that the following statements are equivalent: **10**

i) M is a flat A-module;

ii) M_p is a flat A_p - module for each prime ideal p;

iii) M_m is a flat A_m -module for each maximal ideal m.

b) Let M be a finitely generated A-module, S a multiplicatively closed subset of A. Then prove that $S^{-1}(\text{Ann}(M)) = \text{Ann}\left(S^{-1}M\right)$. **10**

OR

c) Let $A \subseteq B$ be integral domains, A integrally closed, B integral over A. Let $p_1 \supseteq \cdots \supseteq p_n$ be a chain of prime ideals of A, and let $q_1 \supseteq \cdots \supseteq q_m \,(m < n)$ be a chain of prime ideals of B such that $q_i \cap A = p_i \,(1 \le i \le m)$ then prove that the chain $q_1 \supseteq \cdots \supseteq q_m$ can be extended to a chain $q_1 \supseteq \cdots \supseteq q_n$ such that $q_i \cap A = p_i \,(1 \le i \le n)$. **10**

d) Suppose that M has a composition series of length n, Then prove that every composition series of M has length n, and every chain in M can be extended to a composition series. **10**

UNIT – IV

4. a) Prove that in Noetherian ring A very irreducible ideal is primary. **10**

b) Prove that in an Artin ring the nilradical η is nilpotent. **10**

OR

c) Let A be a local domain. Then prove that A is a discrete valuation ring if and only if every non-zero fractional ideal of A is invertible. **10**

d) Prove that the ring of integers in an algebraic number field K is a Dedekind domain. **10**

5. a) Define: ideal quotient and radical of an ideal. **5**

b) Define: Direct Sum and Direct Product of Modules. **5**

c) Explain Field of Fractions of Ring A with respect to multiplicatively closed subset S of a ring. **5**

d) Define Fractional ideal and Artin Ring. **5**

****