M.Sc. - II (Mathematics) New CBCS Pattern Semester-III

PSCMTH12 - Functional Analysis Paper-II

P. Pages : 3

Time : Three Hours

GUG/W/23/13756

Max. Marks : 100

Notes : 1. Solve all **five** questions.
2. Each question carries equal marks.

UNIT – I

1. a) Let P be a real number such that $1 \le P < \infty$. Then prove that the space ℓ_p^n of all n-tuples $x = (x_1, x_2, ..., x_n)$ of scalars, with the norm defined by **10**

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

is a Banach space.

b) Let N and N' be normed linear spaces and T a linear transformation of N into N'. Then prove that the following conditions on T are all equivalent to one another: **10**

i) T is continuous

ii) T is continuous at the origin, in the sense that $x_n \to 0 \Rightarrow T(x_n) \to 0$

iii) There exists a real number $k \ge 0$ with the property that $\|T(x)\| \le k\|x\|$ for every $x \in N$.

OR

c) Let M be a linear subspace of a normed linear space N, and let f be a functional defined on M. If x_0 is a vector not in M, and if **10**

$$M_0 = M + [x_0]$$

is the linear subspace spanned by M and x_0 then prove that f can be extended to a functional f_0 defined on M_0 such that $\|f_0\| = \|f\|$.

d) Prove that if N is a normed linear space and x_0 is a non-zero vector in N, then there exists a functional f_0 in N^* such that $f_0(x_0) = \|x_0\|$ and $\|f_0\| = 1$. **10**

UNIT – II

2. a) State and prove the open mapping theorem. **10**

b) Let B be a Banach space and N a normed linear space. If $\{T_i\}$ is a non – empty set of continuous linear transformations of B into N with the property that $\{T_i(x)\}$ is a bounded subset of N for each vector x in B, then prove that $\{\|T_i\|\}$ is a bounded set of numbers. **10**

OR

c) Let M be a closed linear subspace of a Hilbert space H, let x be a vector not in M, and let d be the distance from x to M. Then prove that there exists a unique vector y_0 in M such that $\|x - y_0\| = d$. **10**

d) Prove that if M is a closed linear subspace of a Hilbert space H, then $H = M \oplus M^{\perp}$. **10**

UNIT – III

3. a) Prove that the adjoint operation $T \to T^*$ on B(H) has the following properties **10**

i) $(T_1 + T_2)^* = T_1^* + T_2^*$

ii) $(\alpha T)^* = \bar{\alpha} T^*$

iii) $(T_1 T_2)^* = T_2^* T_1^*$

iv) $T^{**} = T$

v) $\|T^*\| = \|T\|$

vi) $\|T^* T\| = \|T\|^2$

b) Prove that if A is a positive operator on H, then I + A is non-singular. In particular, show that I + T*T and I + TT* are non-singular for an arbitrary operator T on H. **10**

OR

c) Prove that if T is an operator on H, then T is normal $\Leftrightarrow$ its real and imaginary parts commute. **10**

d) Prove that if $P_1, P_2, ...P_n$ are the projections on closed linear subspace $M_1, M_2, ...M_n$ of H, then $P = P_1 + P_2 + ... + P_n$ is a projection $\Leftrightarrow$ the P_i's are pairwise orthogonal (in the sense that $P_i P_j = 0$ whenever $i \neq j$); and in this case, P is the projection on **10**

$M = M_1 + M_2 + ...M_n$.

UNIT – IV

4. a) Let B be a basis for H, and T an operator whose matrix relative to B is $[\alpha_{ij}]$. Then prove that T is non-singular $\Leftrightarrow [\alpha_{ij}]$ is non-singular, and in this case $[\alpha_{ij}]^{-1} = [T^{-1}]$. **10**

b) Prove that if $B = \{e_i\}$ is a basis for H, then the mapping $T \to [T]$, which assigns to each operator T its matrix relative to B, is an isomorphism of the algebra B(H) onto the total matrix algebra A_n. **10**

OR

c) Show that an operator T on H is normal $\Leftrightarrow$ its adjoint T* is a polynomial in T. **10**

d) Prove that if T is normal, then the eigen spaces M_i's corresponding to the eigen values λ_i's of T span H. **10**

5. a) Show that the linear space $\not\subset$ of the complex numbers with the norm of a number x defined by $\|x\| = |x|$ is a normed linear space. **5**

b) Prove that every non-zero Hilbert space contains a complete orthonormal set. **5**

c) If N is a normal operator on H then prove that $\left\|N^2\right\| = \|N\|^2$. **5**

d) Prove that if T is normal, then the, M_i's (eigen spaces of T corresponding to eigen values λ_i of T) are pairwise orthogonal. **5**

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