M.Sc.-I (Mathematics) (New CBCS Pattern) Semester - II

PSCMTH07 - Lebesgue Measure Theory

P. Pages : 2 | **GUG/S/23/13747**

Time : Three Hours | Max. Marks : 100

* 2 4 8 3 *

Notes : 1. Solve all **five** questions.
2. All questions carry equal marks.

UNIT – I

1. a) Prove that the outer measure of an interval is its length. **10**

b) Let $\langle E_i \rangle$ be a sequence of measurable sets. Then show that $m(UE_i) \le \sum {}_m E_i$. If the sets E_n are pairwise disjoint, then prove that $m(UE_i) = \sum mE_i$. **10**

OR

c) Let $\langle E_n \rangle$ be an infinite decreasing sequence of measurable sets. Let ${}_m E_1$ be finite then prove that $m\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} mE_n$. **10**

d) Let C be a constant and f and two measurable real-valued functions, defined on the same domain. Then prove that the functions f+c, cf, f+g and fg are also measurable. **10**

UNIT – II

2. a) Prove that : If ϕ and Ψ are simple functions which vanish outside a set of finite measure. Then $\int (a\phi + b\Psi) = a\int \phi + b\int \Psi$ and if $\phi \ge \Psi$a.e., then $\int \phi \ge \int \Psi$ **10**

b) State and prove Bounded convergence theorem. **10**

OR

c) Let f be a non negative function which is integrable over a set E. Then prove that for given $\in > 0$ there is a $\delta > 0$ such that for every set A C E with mA $< \delta$ and $\int_A f < \in$. **10**

d) State and prove Lebesgue convergence theorem. **10**

UNIT – III

3. a) Let E be a set of finite outer measure and I a collection of intervals that cover E in the sense of Vitali. Then prove that for given $\in > 0$, there is a finite disjoint collection $\{I_1, ---, I_N\}$ of intervals in I such that **10**

$$m^*\left[E \sim \bigcup_{n=1}^{\infty} I_n\right] < \in .$$

b) Prove that : If f is integrable on [a,b], then the function F defined by **10**

$F(x)=\int_a^x f(t)dt$ is a continuous function of bounded variation on [a,b].

OR

c) If F is bounded and measurable on [a,b] and **10**

$$F(x)=\int_a^x f(t)dt+F(a)$$

Then prove that $F'(x)=f(x)$ for almost all x in [a,b]

d) Prove that a function F is an indefinite integral if and only if it is absolutely continuous. **10**

UNIT – IV

4. a) State and prove Minkowski inequality. **10**

b) Prove that a normed linear space X is complete if and only if every absolutely summable series is summable. **10**

OR

c) Prove that L^∞ is complete. **10**

d) State and prove Riesz representation theorem. **10**

5. a) Prove that if $m^*A=0$, then $m^*(A\cup B)=m^*B$. **5**

b) Show that if **5**

$$f(x)=\begin{cases}0 & x \quad \text{irrational}\\ 1 & x \quad \text{rational}\end{cases}$$

then

$R\int_a^{-b} f(x)=b-a$ and $R\int_{-a}^{b} f(x)dx=0.$

c) Let ϕ be a convex function on $(-\infty,\infty)$ and F an integrable function on [0,1], then prove that $\int\phi(f(t))dt\ge\phi\left[\int f(t)dt\right]$. **5**

d) Show that $\|f+\delta\|_\infty\le\|f\|_\infty+\|\delta\|_\infty$. **5**

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