M.Sc.(Mathematics) (New CBCS Pattern) Semester - II

PSCMTH06 - Field Theory

P. Pages : 2

Time : Three Hours

GUG/S/23/13746

Max. Marks : 100

Notes : 1. Solve all the **five** questions.
2. Each question carries equal marks.

UNIT - I

1. a) Let R be UFD, and $a, b \in R$. Then prove that there exists a greatest common divisor of a & b that is uniquely determined to within an arbitrary unit factor. **10**

b) Prove that every Euclidean domain is a P. I. D. (Principal Ideal Domain). **10**

OR

c) Let R be a unique factorization domain. Then prove that the polynomial ring R[x] over R is also a unique factorization domain. **10**

d) If f(x), g(x) $\in$ R[x], then prove that C(fg) = C(f) C(g) where R is a UFD. **10**

UNIT – II

2. a) Let $f(x) = a_0 + a_1 x + ----+ a_{n-1} x^{n-1} + x^n \in \mathbb{Z}[x]$ be a monic polynomial. If f(x) has a root $a \in \mathbb{Q}$, then prove that $a \in \mathbb{Z}$ & $a \mid a_0$. **10**

b) Let $F \subseteq E \subseteq K$ be fields. If $[K:E] < \infty$ and $[E:F] < \infty$, then prove that **10**

i) $[K:F] < \infty$.

ii) $[K:F] = [K:E][E:F]$.

OR

c) State & prove Eisenstein criterion. **10**

d) Let E be an algebraic extension of F, and let 6 : $E \to E$ be an embedding of E into itself over F. Then prove that 6 is onto and, hence, an automorphism of E. **10**

UNIT – III

3. a) Let P be prime. Then prove that $f(x) = x^P - 1 \in Q[x]$ has splitting field $Q(\alpha)$, where $\alpha \neq 1$ & $\alpha^P = 1$ Also, $[Q(\alpha) : Q] = P - 1$. **10**

b) Let E be a finite extension of a field F. Then prove that the following are equivalent. **10**

a) $E = F(\alpha)$ for some $\alpha \in E$.

b) There are only a finite number of intermediate fields between F & E.

OR

c) If $f(x) \in F[x]$ is irreducible over F, then prove that all roots of f(x) have the same multiplicity. **10**

d) If the multiplicative group F* of nonzero elements of a field F is cyclic, then prove that F is finite. **10**

UNIT – IV

4. a) Let E be a finite separable extension of a field F then prove that the following are equivalent: **10**

i) E is a normal extension of F.

ii) F is the fixed field of G(E|F).

iii) [E:F] = | G(E|F)|

b) Prove that the Galois group of $x^4 - 2 \in Q[x]$ is the octic group (= group of symmetries of a square). **10**

OR

c) Prove that every polynomial $f(x) \in \mathbb{C}(x)$ factors into linear factors in $\mathbb{C}(x)$. **10**

d) Prove that the Galois group of $x^4 + 1 \in Q[x]$ is the Klein four-group. **10**

5. a) Show that 3 is irreducible but not prime in the ring $\mathbb{Z}\left[\sqrt{-5}\right]$. **5**

b) Find the minimal polynomials over Q of the following numbers. **5**

i) $\sqrt{2} + 5$ ii) $3\sqrt{2} + 5$

c) Let $F = \mathbb{Z}/(2)$. Prove that the splitting field of $x^3 + x^2 + 1 \in F[x]$ is a finite field with eight elements. **5**

d) Let F be a field of characteristic $\neq 2$. Let $x^2 - a \in F[x]$ be an irreducible polynomial over F. Then prove that its Galois group is of order 2. **5**

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