M.Sc. (Mathematics) New CBCS Pattern Semester-I

PSCMTH05B - Core Elective - Ordinary Differential Equations

P. Pages : 4 | GUG/W/23/13742

Time : Three Hours | Max. Marks : 100

Notes : 1. Solve all **five** questions.
2. Each question carries equal marks.

UNIT – I

1. a) Suppose a and b are continuous functions on an interval I. Let A be a function such that $A' = a$. Then prove that the function ψ given by **10**

$$\psi(x) = e^{-A(x)} \int_{x_0}^{x} e^{A(t)} b(t)\, dt,$$

Where x_0 is in I, is a solution of the equation.

$y' + a(x)y = b(x)$

on I. Also, prove that the function ϕ_1 given by

$\phi_1(x) = e^{-A(x)}$

is a solution of the homogeneous equation

$y' + a(x)y = 0$

Next, show that if C is any constant,

$\phi = \psi + C\phi_1$ is a solution of

$y' + a(x)y = b(x)$

and every solution of this differential equation has this form.

b) Consider the equation **10**

$Ly' + Ry = E$

Where L, R, E are positive constants.

i) Solve this equation

ii) Find the solution ϕ satisfying, $\phi(0) = I_0$, where I_0 is a given positive constant.

iii) Show that every solution tends to E/R as $x \to \infty$.

OR

c) Prove that: **10**

Two solutions ϕ_1, ϕ_2 of $L(y) = 0$ are linearly independent on an interval I, if and only if

$w(\phi_1, \phi_2)(x) \neq 0$

for all x in I.

d) Compute the solution ψ of $y''' + y'' + y' + y = 1$ which satisfies $\psi(0) = 0, \psi'(0) = 1,$ $\psi''(0) = 0$. **10**

UNIT – II

2. a) Let $b_1,......,b_n$ be non-negative constants such that for all x in I. **10**

$$|a_j(x)| \leqq b_j, \quad (j=1,......,n)$$

and define K by

$$K = 1 + b_1 + ... + b_n$$

If x_0 is a point in I, and ϕ is a solution of

$$L(y) = y^{(n)} + a_1(x)\,y^{(n-1)} + ... + a_n(x)\,y = 0$$

on I, then prove that

$$\|\phi(x_0)\| e^{-k|x-x_0|} \leqq \|\phi(x)\| \leqq \|\phi(x_0)\| e^{k|x-x_0|}$$

for all x in I.

b) Consider the equation **10**

$$L(y) = y'' + a_1(x)y + a_2(x)y = 0\ ,$$

Where a_1, a_2 are continuous on some interval I. Show that a_1 and a_2 are uniquely determined by any basis ϕ_1, ϕ_2 for the solutions of $L(y) = 0$. show that

$$a_1 = \frac{\begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1'' & \phi_2'' \end{vmatrix}}{w(\phi_1, \phi_2)},\ a_2 = \frac{\begin{vmatrix} \phi_1' & \phi_2' \\ \phi_1'' & \phi_2'' \end{vmatrix}}{w(\phi_1, \phi_2)}$$

OR

c) Let b be continuous as an interval I, and let $\phi_1, \phi_2, ..., \phi_n$ be a basis for the solutions of **10**
$L(y) = y^{(n)} + a_1(x)y^{(n-1)} + ... + a_n(x)\,y = 0$ on I, where $a_1, a_2, ..., a_n$ are continuous functions on an interval I. Then prove that every solution of

$$L(y) = y^{(n)} + a_1(x)\,y^{(n-1)} + ... + a_n(x)y = b(x)$$

Can be written as

$$\psi = \psi_p + C_1\phi_1 + ... + C_n\phi_n\ ,$$

Where ψ_p is a particular solution of $L(y) = b(x)$ and $C_1, C_2, ..., C_n$ are constants. Also, prove that every such ψ is a solution of $L(y) = b(x)$ and a particular solution ψ_p is given by

$$\psi_p(x) = \sum_{k=1}^{n} \phi_k(x) \int_{x_0}^{k} \frac{w_k(t)b(t)}{w(\phi_1, ..., \phi_n)(t)}\, dt$$

d) Find all solutions of the equation **10**

$$x^2y'' + xy' + 4y = 1$$

for $|x| > 0$.

UNIT – III

3. a) Let M, N be two real – valued functions which have continuous first partial derivatives on some rectangle. **10**

$$R : |x - x_0| \le a, \ |y - y_0| \le b$$

Then prove that the equation

$$M(x, y) + N(x, y)\, y' = 0$$

is exact in R, if and only if,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

in R.

b) Prove that : A function ϕ is a solution of the initial value problem. **10**

$$y' = f(x, y), \ y(x_0) = y_0$$

on an interval I if and only if it is a solution of the integral equation

$$y = y_0 + \int_{x_0}^{x} f(t, y)\, dt$$

on I.

OR

c) Let f be a real-valued continuous function on the strip **10**

$$S : |x - x_0| \le a, \ |y| < \infty, (a > 0)$$

and suppose that f satisfies on S a Lipschitz condition with constant $k > 0$. Then prove that the successive approximations $\{\phi_k\}$ for the problem.

$$y' = f(x, y), \ y(x_0) = y_0$$

exist on the entire interval $|x - x_0| \le a$, and converge there to a solution of this initial value problem.

d) Let f, g be continuous on R, and suppose f satisfies a Lipschitz condition there with Lipschitz constant K. Let ϕ, ψ be solutions of the two initial value problems **10**

$$y' = f(x, y), \ \ y(x_0) = y_1,$$

$$y' = g(x, y), \ \ y(x_0) = y_2,$$

(where f, g are both continuous real – value – d functions on

$$R : |x - x_0| \le a, | \, y - y_0| \le b, \ (a, b > 0)$$

and $(x_0, y_1), (x_0, y_2)$ are points in R) respectively on an interval I containing x_0, with graphs contained in R. Then for non-negative constants $\in, \delta$, if the inequalities.

$$|f(x, y) - g(x, y)| \le \in \ ((x, y) \text{ in } R)$$

and $|y_1 - y_2| \le \delta$

are valid, then

$$|\phi(x) - \psi(x)| \le \delta\, e^{k|x - x_0|} + \frac{\in}{k}(e^{k|x - x_0|} - 1)$$

for all x in I.

UNIT – IV

4. a) Find a solution ϕ of **10**

$$y'' = -\frac{1}{2y^2}$$

Satisfying $\phi(0) = 1,\ \phi'(0) = -1.$

b) For each $y = (y_1, y_2, ..., y_n)$ in C_n let **10**

$$\|y\| = (y_1, \bar{y}_1 + ... + y_n, \bar{y}_n)^{1/2}$$

The positive square root being understood. This is the Euclidean length of y.

i) Show that $\|y\| \le |y| \le \sqrt{n}\,\|y\|$

ii) Show that a sequence $\{y_m\}, (m = 1, 2, ...)$ of vectors in C_n is such that

$$|y_m - y| \to 0,\ (m \to \infty)$$

if and only if

$$\|y_m - y\| \to 0,\ (m \to \infty)$$

OR

c) Let f be the vector – valued function defined on **10**

$$R : |x| \le 1,\ |y| \le 1,\ (y \text{ in } C_2)$$

by $f(x, y) = (y_2^2 + 1, x + y_1^2)$.

i) Find an upper bound M for $|f(x, y)|$ for (x, y) in R.

ii) Compute a Lipschitz constant k for f on R.

d) Consider the system **10**

$$y_1' = 3y_1 + xy_3,$$

$$y_2' = y_2 + x^3 y_3,$$

$$y_3' = 2xy_1 - y_2 + e^x y_3$$

Show that every initial value problem for this system has a unique solution which exists for all real x.

5. a) Find all solutions of the equation **5**

$$y' + y = e^x$$

b) Find all solution of the equation **5**

$$x^2 y'' + 2xy' - 6y = 0$$

For $x > 0$

c) Determine whether the equation **5**

$$2xy\,dx + (x^2 + 3y^2)\,dy = 0$$

is exact for $(x, y) \in R^2$, and solve it.

d) Solve the equation **5**

$$y'' = yy''$$

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