M.Sc.- I (Mathematics) New CBCS Pattern Semester-I

PSCMTH04 : Linear Algebra

P. Pages : 2 **GUG/W/23/13740**

Time : Three Hours Max. Marks : 100

Notes : 1. Solve all **five** questions.
2. Each questions carry equal marks.

UNIT – I

1. a) Prove that the span of any subset S of a vector space V is a subspace of V. Moreover, prove that any subspace of V that contains S must also contain the span of S. **10**

b) If a vector space is generated by a finite set S, then prove that some subset of S is a basis for V. **10**

OR

c) Let W be a subspace of a finite dimensional vector space V. Then prove that W is finite dimensional & $\dim(W) \le \dim(V)$. Prove that if $\dim(W) = \dim(V)$, then V = W. **10**

d) Let S be a linearly independent subset of a vector space V. Prove that there exists a maximal linearly independent subset of V that contains S. **10**

UNIT – II

2. a) Let V & W be vector spaces & $T: V \to W$ be linear. Then prove that null space N(T) of T and range R(T) of T are subspaces of V & W, respectively. **10**

b) Let V & W be finite dimensional vector spaces with ordered bases β & γ, respectively. **10**
Let $T: V \to W$ be linear. Then prove that T is invertible iff $[T]_\beta^\gamma$ is invertible

Furthermore, prove that $\left[T^{-1}\right]_\gamma^\beta = \left([T]_\beta^\gamma\right)^{-1}$.

OR

c) Let V & W be vector spaces, & let $T: V \to W$ be linear. If V is finite-dimensional, then prove that nullity (T) + rank (T) = dim (V). **10**

d) Let V & W be finite-dimensional vector spaces over the same field. Then prove that V is isomorphic to W iff dim (V) = dim (W). **10**

UNIT – III

3. a) Find all the eigen vectors of the matrix $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$ & prove that A is diagonalizable. **10**

b) State & prove Cayley – Hamilton theorem. **10**

OR

c) Let T be a linear operator on a vector space V, and let $\lambda_1, \lambda_2, ..., \lambda_k$ be distinct eigen values of T. If $V_1, V_2, ..., V_k$ are eigen vectors of T such that λ_i corresponds to V_i $(1 \le i \le k)$, then prove that $\{V_1, V_2, ..., V_k\}$ is linearly independent. **10**

d) Let T be a linear operator on a finite-dimensional vector space V, & let λ be an eigenvalue of T having multiplicity m. Then prove that $1 \le \dim(E_\lambda) \le m$. **10**

UNIT – IV

4. a) Let V be an inner product space & $S = \{W_1, W_2, ..., W_n\}$ be a linearly independent subset of V. Define $S' = \{V_1, V_2, ..., V_n\}$, where $V_1 = W_1$ and $V_k = W_k - \sum_{j=1}^{k-1} \frac{\langle W_k, V_j \rangle}{\| V_j \|^2}$ For $2 \le k \le n$. Then prove that is an orthogonal set of nonzero vectors such that $\text{span}(S') = \text{span}(S)$. **10**

b) Let V be a finite-dimensional inner product space over F, & let $g : V \to F$ be a linear transformation. Then prove that there exists a unique vector $y \in V$ such that $g(x) = \langle x, y \rangle$ for all $x \in V$. **10**

OR

c) Let T be a linear operator on a finite-dimensional vector space V, & let λ be an eigen value of T. Then prove that k_λ has an ordered basis consisting of a union of disjoint cycles of generalized eigen vectors corresponding to λ. **10**

d) Let T be a linear operator on a finite-dimensional inner product space V. Suppose that the characteristic polynomial of T splits. Then prove that there exists an orthonormal basis β for V such that the matrix $[T]_\beta$ is upper triangular. **10**

5. a) Define linearly dependent set & basis. **5**

b) Define invertible linear transformation & null space. **5**

c) Find the eigenvalues of the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ **5**

d) Define : **5**

i) Unitary operator

ii) The adjoint of a linear operator.

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