M.Sc. - I (Mathematics) New CBCS Pattern Semester-I

PSCMTH02 - Real Analysis

P. Pages : 2 **GUG/W/23/13738**

Time : Three Hours * 7 3 2 4 * Max. Marks : 100

Notes : 1. Solve all **five** questions.
2. Each question carries equal marks.

UNIT – I

1. a) Prove that the sequence of functions $\{f_n\}$ defined on E, converges uniformly on E iff for every $\in > 0$ there exists an integer N such that $m \geq N,\ n \geq N,\ x \in E$ implies $|f_n(x) - f_m(x)| \leq \in$. **10**

b) Suppose $\{f_n\}$ is a sequence of functions, differentiable on [a, b] and such that $\{f_n(x_0)\}$ converges for some point x_0 on [a, b]. If $\{f_n'\}$ converges uniformly on [a,b], then prove that $\{f_n\}$ converges uniformly on [a, b] to a function f, and $f'(x) = \lim_{n\to\infty} f_n'(x)\ (a \leq x \leq b)$. **10**

OR

c) Let α be monotonically increasing on [a,b]. Suppose $f_n \in R(\alpha)$ on [a, b], for n = 1, 2, 3,----, and suppose $f_n \to f$ uniformly on [a, b]. Then prove that $f \in R(\alpha)$ on [a,b], and $\int_a^b f\ d\alpha = \lim_{n\to\infty} \int_a^b f_n d\alpha$. **10**

d) State & prove the stone-Weier – strass theorem. **10**

UNIT – II

2. a) If X is a complete metric space, and if ϕ is a contraction of X into X, then prove that there exists one and only one $x \in X$ such that $\phi(x) = x$. **10**

b) Suppose f is a τ' - mapping of an open set $\in \subset \mathrm{lR}^n$ into lR^n, F'(a) is invertible for some $a \in e$ and b = f(a). Then prove that **10**

a) there exist open sets $U \& V$ in lR^n such that $a \in U,\ b \in V,$ F is one-to-one on U, and f(U) = V;

b) If g is the inverse of F, defined in V by $g(f(x)) = x\ \ (x \in U)$, then $g \in \tau'(V)$.

OR

c) Suppose F maps an open set $E \subset \mathrm{lR}^n$ into lR^m. Then prove that $f \in \xi'(E)$ if & only if the partial derivatives D_jF_i exist and are continuous on E for $1 \leq i \leq m,\ 1 \leq j \leq n$. **10**

d) Suppose E is an open set in lR^n, F maps E into lR^m, F is differentiable at $x_0 \in E$, g maps an open set containing F(E) into lR^k, and g is differentiable at $F(x_0)$. Then prove that the mapping F of E into lR^k defined by $F(x) = g(f(x))$ is differentiable at x_0, and $F'(x_0) = g'(f(x_0)).f'(x_0)$. **10**

UNIT – III

3. a) Prove that $GL(n, \mathbb{R})$ & $GL(n, \mathbb{C})$ are manifold of dimension n^2 & $2n^2$ respectively. **10**

b) Prove that any atlas $u = \{(U_\alpha, \phi_\alpha)\}$ on a locally Euclidean space is contained in a unique maximal atlas. **10**

OR

c) Consider S' as the unit circle in the real plane $\mathbb{R}^2$ with defining equation $x^2 + y^2 = 1$, and describe a C^∞ atlas with four charts on it. **10**

d) Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas on a locally Euclidean space. If two charts (V, Ψ) and (W, δ) are both compatible with the atlas $\{(U_\alpha, \phi_\alpha)\}$, then prove that they are compatible with each other. **10**

UNIT – IV

4. a) Let M & N be manifolds and $\pi : MxN \to M$, $\pi(p,q) = p.$ the projection to the first factor. Prove that π is a C^∞ map. **10**

b) If (U, ϕ) is a chart on a manifold M of dimension n, then prove that the co-ordinate map $\phi : U \to \phi(U) \subset \mathbb{R}^n$ is a diffeomorphism. **10**

OR

c) Suppose $F : N \to M$ is C^∞ at $P \in N$. If (U, ϕ) is any chart about P in N and (V, Ψ) is any chart about F(P) in M, then prove that $\Psi \circ F \circ \phi^{-1}$ is C^∞ at $\phi(P)$. **10**

d) Let M be a manifold of dimensions n, and $F : M \to \mathbb{R}$ a real valued function on M. Prove that the following are equivalent :- **10**

i) The function $F : M \to \mathbb{R}$ is C^∞.

ii) The manifold. M has an atlas such that for every chart (U, ϕ) in the atlas, $F \circ \phi^{-1} : \mathbb{R}^n \supset \phi(U) \to \mathbb{R}$ is C^∞.

iii) For every chart (V, Ψ) on M, the function $F \circ \Psi^{-1} : \mathbb{R}^n \supset \Psi(V) \to \mathbb{R}$ is C^∞.

5. a) Define **5**

i) Equicontinuous family

ii) Pointwise bounded & uniformly bounded sequence of functions.

b) Define **5**

i) Differentiable function ii) Partial derivatives

c) Define : **5**

i) Locally Euclidean of dimension n topological space.

ii) Atlas

d) If $F : N \to M$ & $G : M \to P$ are C^∞ maps of manifolds, then prove that the composite $G \circ F : N \to P$ is C^∞. **5**

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