M.Sc. - I (Mathematics) New CBCS Pattern Semester-I

PSCMTH01 - Group Theory & Ring Theory

P. Pages : 2 | **GUG/W/23/13737**

Time : Three Hours | Max. Marks : 100

Notes : 1. Solve all **five** questions.
2. Each questions carries equal marks.

UNIT – I

1. a) Prove that every group is isomorphic to a permutation group. **10**

b) Let G be a group, and let G' be the derived group of G. Then prove that **10**

i) $G' \triangleleft G$

ii) G / G' is abelian

iii) If $H \triangleleft G$ then G/H is abelian iff $G' \subset H$.

OR

c) Prove that a nonabelian group of order σ is isomorphic to S_3. **10**

d) Let G be a finite group acting on a finite set X. Then the number K of orbits in X under G is $K = \frac{1}{|G|} \sum_{g \in G} |X_g|$. **10**

UNIT – II

2. a) Let G be a nilpotent group. Then prove that every subgroup of G and every homomorphic image of G are nilpotent. **10**

b) If $\alpha, \sigma \in S_n$, then $\tau = \alpha \sigma \alpha^{-1}$ is the permutation obtained by applying α to the symbols in σ. Hence prove that any two conjugate permutations in S_n have the same cycle structure. Conversely, prove that any two permutations in S_n with the same cycle structure are conjugate. **10**

OR

c) Prove that any two composition series of a finite group are equivalent. **10**

d) Prove that a group G is nilpotent iff G has a normal series **10**

$\{e\} = G_0 \subset G_1 \subset .... \subset G_m = G$

Such that $G_i / G_{i-1} \subset Z(G / G_{i-1})$ for all $i = 1, 2, ..., m$.

UNIT – III

3. a) Let A be a finite abelian group, and let P be a prime. If P divides $|A|$, then prove that A has an element of order P. **10**

b) Prove that there are no simple groups of orders 63, 56 and 36. **10**

OR

c) Let G be a group of order pq, where p & q are prime numbers such that $p > q$ & $q \nmid (p-1)$. Then prove that G is cyclic. **10**

d) Let G be a finite group, and let p be a prime. Then prove that all Sylow p – subgroups of G are conjugate, & their number n_p divides O(G) & satisfies $n_p \equiv 1 (\mathrm{mod}\, p)$. **10**

UNIT – IV

4. a) If a ring R has unity, then prove that every ideal I in the matrix ring R_n is of the form A_n, where A is some ideal in R. **10**

b) Let f be a homomorphism of a ring R into a ring S with kernel N. Then prove that $R / N \simeq \mathrm{Im} f$ **10**

OR

c) For any two ideals A & B in a ring R, prove that **10**

i) $$\frac{A+B}{B} \simeq \frac{A}{A \cap B}$$

ii) $$\frac{A+B}{A \cap B} \simeq \frac{A+B}{A} \times \frac{A+B}{B} \simeq \frac{B}{A \cap B} \times \frac{A}{A \cap B}$$

d) Prove that in a nonzero commutative ring with unity, an ideal M is maximal iff R/M is a field. **10**

5. a) Let G be a group & H < G of finite index n. Then prove that there is a homomorphism $\phi : G \to S_n$ such that $\ker \phi = \bigcap\limits_{x \in G} x H x^{-1}$. **5**

b) Prove that the derived group of S_n is A_n. **5**

c) If the order of group is 42, prove that its Sylow 7, subgroup is normal. **5**

d) Define Maximal & Prime ideals. **5**

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