M.Sc. (Mathematics) (NEW CBCS Pattern) Sem-IV

PSCMTH16: Dynamical Systems

P. Pages : 2 **GUG/W/22/13767**

Time : Three Hours Max. Marks : 100

Notes : 1. Solve all **five** questions.
2. Each question carries equal marks.

UNIT – I

1. a) Let the function $f : w \to E$ be c^1. Then prove that f is locally Lipschitz. **10**

b) Let $W \subset E$ be open and suppose $f : W \to E$ has Lipschitz constant K. Let y (t), z (t) be solutions to $x^1 = f(x)$ on the closed interval $[t_0, t_1]$. Then prove that for all $t \in [t_0, t_1]$: $|y(t) - z(t)| \le |y(t_0) - z(t_0)| \exp(k(t - t_0))$. **10**

OR

c) Let a c^1 map $f : W \to E$ be given. Suppose two solutions u (t), v (t) of $x^1 = f(x)$ are defined on the same open interval J containing t_0 and satisfy $u(t_0) = v(t_0)$. Then prove that $u(t) = v(t)$ for all $t \in J$. **10**

d) Prove that $\oint_t$ sends U on to an open set V and $\oint_{-t}$ is defined on V and sends V onto U. The composition $\oint_{-t} \oint_t$ is the identity map of U, the composition $\oint_t \oint_{-t}$ is the identity map of V. **10**

UNIT – II

2. a) Let $W \subset E$ be open and $f : W \to E$ continuously differentiable. Suppose $f(\bar{x}) = 0$ and $\bar{x}$ is a stable equilibrium point of the equation $x^1 = f(x)$. Then prove that no eign value of $Df(\bar{x})$ has positive real part. **10**

b) Prove that E* is isomorphic to E and thus has the same dimension. **10**

OR

c) Let E be a real vector space with an inner product and let T be a self adjoint operator on E. Then prove that the eigen values of T are real. **10**

d) Let E be a real vector space with an inner product. Then prove that any self-adjoint operator on E can be diagonalized. **10**

UNIT – III

3. a) Prove that a non empty compact limit set of a C^1 planar dynamical system, which contains no equilibrium point, is a closed orbit. **10**

b) Define a limit cycle. Let γ be a closed orbit enclosing an open set U contained in the domain w of the dynamical system. Then prove that U contains an equilibrium. **10**

OR

c) Show that every trajectory of the Volterra-Lotka equation $x' = (A - By)x,\ y' = (C_x - D)y$, where A, B, C, D > 0 is a closed orbit. (except the equilibrium z and the coordinate axes) **10**

d) Prove that **10**

The flow $\oint_t$ of $x' = M(x, y)x,\ y' = N(x, y)y$, where the growth rates M and N are C^1 functions of non negative variables x, y has the following property: for all $p = (x, y), x \geq 0,\ y \geq 0$, the limit $\lim_{t \to \infty} \oint_t (p)$ exists & is one of a finite number of equilibria.

UNIT – IV

4. a) Let $\bar{x}$ be a fixed point of a discrete dynamical system $g : W \to E$. If the eigen values of $Dg(\bar{x})$ are less than 1 in absolute value, Then prove that $\bar{x}$ is asymptotically stable. **10**

b) Let γ be an asymptotically stable closed orbit of period λ. Then prove that γ has a neighborhood $U \subset W$ such that every point of U has asymptotic period λ. **10**

OR

c) Let $W \subset E$ be open and let $f : W \to E$ be $C^r, 1 \leq r \leq \infty$. Then prove that the flow $\oint : \Omega \to E$ of the differential equation $x' = f(x)$ is also C^r. **10**

d) Assume E is normed. Let $r > \left\| Df(x_0)^{-1} \right\|$. Let $V \subset W$ be an open ball around x_0 such that **10**

$\left\| Df(y)^{-1} \right\| < \nu$ and

$\left\| Df(y) - Df(z) \right\| < \frac{1}{r}$ for all $y, z \in V$

Then prove that F/V is one-to-one.

5. a) Explain the flow of a differential equation. **5**

b) Define- **5**

i) Stability ii) Asymptotic stability iii) Instability

iv) Equilibrium point v) Sink

c) If x and z are on the same trajectory, then show that $L_w(x) = L_w(z)$, similarly for $\alpha -$ limits. **5**

d) Explain asymptotic stability of closed orbits. **5**

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