M.Sc.(Mathematics) (New CBCS Pattern) Semester - III

PSCMTH12 - Functional Analysis

P. Pages : 2 **GUG/S/23/13756**

Time : Three Hours Max. Marks : 100

Notes : 1. All **five** questions are compulsory.
2. Each question carries equal marks.

UNIT – I

1. a) Let M be a linear subspace of a normed linear space N, and let f be a functional defined on M. If x_0 is a vector not in M, and if $M_0 = M + [x_0]$ is the linear subspace spanned by M & x_0, then prove that f can be extended to a functional f_0 defined on M_0 such that $\| f_0 \| = \| f \|$. **10**

b) If M is a closed linear subspace of a normed linear space N and x_0 is a vector not in M, then prove that there exists a functional f_0 in N^* such that $f_0(M) = 0$ and $f_0(x_0) \neq 0$. **10**

OR

c) Let N & N' be normed linear spaces and T a linear transformation of N into N'. Then prove that the following conditions on T are all equivalent to one another: **10**

i) T is continuous;

ii) T is continuous at the origin, in the sense that $x_n \to 0 \Rightarrow T(x_n) \to 0$;

iii) There exists a real number $K \geq 0$ with the property that $\| T(x) \| \leq K \| x \|$ for every $x \in N$;

iv) If $S = \{ x : \|x\| \leq \perp \}$ is the closed unit sphere in N, then its image $T(S)$ is a bounded set in N'

d) Let M be a linear subspace of a normed linear space N, and let F be a functional defined on M. Then prove that F can be extended to a functional F_o defined on the whole space N such that $\|F_0\| = \|F\|$ **10**

UNIT – II

2. a) Prove that a closed convex subset C of a Hilbert space. H contains a unique vector of smallest norm. **10**

b) If M is a proper closed linear subspace of a Hilbert space H, then prove that there exists a non - zero vector Z_o in H such that $Z_o \perp M$ **10**

OR

c) If B & B' are Banach spaces and if T is a linear transformation of B into B', then prove that T is continuous iff its graph is closed. **10**

d) If M & N are closed linear subspaces of a Hilbert space H such that $M \perp N$, then prove that the linear subspace M + N is also closed. **10**

UNIT – III

3. a) Prove that the adjoint operation $T \to T^*$ on $B(H)$ has the following properties: **10**

i) $(\alpha T)^* = \bar{\alpha} T^*$

ii) $(T_1 T_2)^* = T_2^* T_1^*$

iii) $\|T^*\| = \|T\|$

iv) $\|T^* T\| = \|T\|^2$

b) If P is a projection on H with range M is null space N, then prove that $M \perp N$ iff P is self - adjoint; and in this case, $N = M^{\perp}$. **10**

OR

c) If T is an operator on H, then prove that T is normal iff its real and imaginary parts commute. **10**

d) If T is an operator on H for which $(Tx, x) = 0$ for all x, then prove that T = 0. **10**

UNIT – IV

4. a) If $B = \{e_i\}$ is a basis for H, then prove that the mapping $T \to [T]$, which assigns to each operator T its matrix relative to B, is an isomorphism of the algebra $B(H)$ onto the total matrix algebra A_n. **10**

b) Prove that two matrices in A_n are similar iff they are the matrices of a single operator on H relative to (possibly) different bases. **10**

OR

c) Let B be a basis for H, and T an operator whose matrix relative to B is $[\alpha_{ij}]$. Then prove that T is non-singular iff $[\alpha_{ij}]$ is non – singular & in this case $[\alpha_{ij}]^{-1} = [T^{-1}]$ **10**

d) State & prove the spectral theorem. **10**

5. a) Define: **5**

i) Bounded linear transformation. ii) Banach space.

b) Define: **5**

i) Orthogonal set. ii) Orthogonal complement of a set.

c) If P is the projection on a closed linear subspace M of H, then prove that M is invariant under an operator $T \Leftrightarrow TP = PTP$. **5**

d) Explain matrix representation of a linear operator on a vector space. **5**

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