M.Sc. (Mathematics) (NEW CBCS Pattern) Sem-II

PSCMTH07: Lebesgue Measure Theory

P. Pages : 2 **GUG/W/22/13747**

Time : Three Hours Max. Marks : 100

* 3 4 5 1 *

Notes : 1. Solve all **five** questions.
2. All questions carry equal marks.

UNIT – I

1. a) If E_1 and E_2 are measurable. Then prove that $E_1 \cup E_2$ is also measurable. **10**

b) Prove that the outer measure of an interval is its length. **10**

OR

c) Prove that : Every Borel set is measurable. In particular, each open set and each closed set is measurable. **10**

d) Prove that if E is a measurable set of finite measure, and $\langle f_n \rangle$ a sequence of measurable functions defined on E. If f is a real valued function such that for each x in E, $f_n(x) \to f(x)$. Then given $\in > 0$ and $\delta > 0$, there is a measurable set $A \subset E$ with $mA < \delta$ and an integer N such that for all $x \notin A$ and all $n \geq N$,
$|f_n(x) - f(x)| < \in$ **10**

UNIT – II

2. a) State and prove the Bounded convergence theorem. **10**

b) Prove that, If f and g are non-negative measurable functions, then. **10**

i) $\int_E cf = c\int_E f, c > 0$

ii) $\int_E f + g = \int_E f + \int_E g$

iii) If $f \leq g$ a. e, then

iv) $\int_E f \leq \int_E g$

OR

c) Let f be a nonnegative function which is integrable over a set E. Then prove that given $\in > 0$ there is a $\delta > 0$ such that for every set $A \subset E$ with $mA < \delta$ we have $\int_A f < \in$. **10**

d) Show that if f is integrable over E, then so is $|f|$ and **10**

$$\left|\int_E f\right| \leq \int_E |f|$$

Does the integrability of $|f|$ imply that of f.

UNIT – III

3. a) Prove that if f is integrable on [a, b], then the function F defined by **10**

$$F(x)=\int_a^x f(t)\,dt$$

is a continuous function of bounded variation on [a, b].

b) Let E be a set of finite outer measure and I a collection of intervals that cover E in the sense of Vitali. **10**

Then prove that, given $\in > 0$, there is a finite disjoint collection $\{I_1, I_2, ----I_N\}$ of intervals in I such that $m^*\left[E \sim \bigcup_{n=1}^{N} I_n\right] < \in$

OR

c) Prove that: If f is integrable on [a, b] and $\int_a^x f(t)\,dt = 0$ for all $x \in [a,b]$, then f(t) = 0 a. e. in [a, b] **10**

d) Prove that a function F is an indefinite integral if and only if it is absolutely continuous. **10**

UNIT – IV

4. a) State and prove Minkowski Inequality for L^p spaces with $1 \le P < \infty$. **10**

b) Let a, b be nonnegative, $1 < p < \infty, \frac{1}{p}+\frac{1}{q}=1$. **10**

Establish Young's inequality.

$$ab \le \frac{a^p}{p}+\frac{b^q}{q}$$

OR

c) Prove that the L^p spaces are complete. **10**

d) Let g be an integrable function on [0, 1] and suppose that there is a constant M such that $\left|\int f_g\right| \le M\|f\|_p$ **10**

For all bounded measurable functions f. Then prove that g is in L^q, and $\|g\|_q \le M$.

5. a) Prove that the set [0, 1] is not countable. **5**

b) Show that if **5**

$$f(x)=\begin{cases}0 & x \text{ irrational}\\ 1 & x \text{ rational}\end{cases}$$

then

$$R\int_a^{-b} f(x)\,dx = b-a \text{ and } R\int_{-a}^{b} f(x)\,dx = 0$$

c) If f is of bounded variation on [a, b], then prove that **5**

$$T_a^b = P_a^b + N_a^b \text{ and } f(b)-f(a) = P_a^b - N_a^b$$

d) Show that if $f \in L^p, g \in L^p$, then $f+g \in L^p$ even for 0 < p < 1. **5**

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