M.Sc. (Mathematics) (NEW CBCS Pattern) Sem-II

PSCMTH06 : Field Theory

P. Pages : 2 **GUG/W/22/13746**

Time : Three Hours Max. Marks : 100

Notes :
1. Solve all five questions.
2. Each question carries equal marks.

UNIT – I

1. a) Prove that every Euclidean domain is a PID. **10**

b) Prove that the product of two primitive polynomials is primitive. **10**

OR

c) Prove that every PID is a UFD, but a UFD is not necessarily a PID. **10**

d) If R is a UFD, then prove that the factorization of any element in R as a finite product of irreducible factors is unique to within order & unit factors. **10**

UNIT – II

2. a) Let $f(x) \in \mathbb{Z}[x]$ be primitive. Then prove that $f(x)$ is reducible over Q if & only if $f(x)$ is reducible over $\mathbb{Z}$. **10**

b) Let $F \subseteq E \subseteq K$ be fields. If $[K:E] < \infty \text{ and } [E:F] < \infty$, then prove that **10**

i) $[K:F] < \infty$

ii) $[K:F] = [K:E].[E:F]$

OR

c) Let $f(x) = a_0 + a_1 x + ----- + a_n x^n \in \mathbb{Z}[x], n \geq 1$. If there is a prime p such that $p^2 \nmid a_0, p \mid a_0, p \mid a_1 ----, p \mid a_{n-1}, p \nmid a_n$, then prove that f(x) is irreducible over Q. **10**

d) Let $p(x)$ be an irreducible polynomial in $F[x]$. Then prove that there exists an extension E of F in which $p[x]$ has a root. **10**

UNIT – III

3. a) Prove that the degree of the extension of the splitting field of $x^3 - 2 \in Q[x]$ is 6. **10**

b) Let E be a finite extension of a field F. The prove that the following are equivalent. **10**

a) $E = F(\alpha)$ for some $\alpha \in E$.

b) There are only a finite number of intermediate fields between F & E.

OR

c) Let p be prime. Then prove that $f(x) = x^p - 1 \in Q[x]$ has splitting field $Q(\alpha)$, where $\alpha \neq 1 \,\&\, \alpha^p = 1$. Also, prove that $[Q(\alpha) : Q] = p - 1$ **10**

d) If E is a finite separable extension of a field F, then prove that E is a simple extension of F. **10**

UNIT – IV

4. a) Prove that the Galois group of $x^4 + 1 \in Q[x]$ is the Klein four – group. **10**

b) Prove that every polynomial $f(x) \in \subset [x]$ factors into linear factors in $\subset [x]$ **10**

OR

c) Let E be a finite separable extension of a field F. Then prove that the following are equivalent: **10**

i) E is a normal extension of F

ii) F is the fixed field of $G(E/F)$.

iii) $[E : F] = |G(E/F)|$

d) Prove that the Galois group of $x^4 - 2 \in Q[x]$ is the octic group. **10**

5. a) Define **5**

i) Irreducible element

ii) Prime element.

b) Show that $x^3 + 3x + 2 \in \mathbb{Z}/(7)[x]$ is irreducible over the field $\mathbb{Z}/(7)$ **5**

c) Define **5**

i) Splitting field

ii) Normal extension

d) Prove that the group $G(Q(\alpha)/Q)$, where $\alpha^5 = 1 \,\&\, \alpha \neq 1$, is isomorphic to the cyclic group of order 4. **5**

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