M.Sc. (Mathematics) (NEW CBCS Pattern) Sem-I

PSCMTH05 (B) : Core Elective Course : Ordinary Differential Equations

P. Pages : 4 | GUG/W/22/13742

Time : Three Hours | Max. Marks : 100

Notes : 1. Solve all **five** questions.
2. Each question carries equal marks.

UNIT – I

1. a) Consider the equation **10**

$$y' + ay = b(x),$$

where a is a constant, and b is a continuous function on an interval. If x_0 is a point in I and C is any constant, then prove that the function ϕ defined by

$$\phi(x) = e^{-ax} \int_{x_0}^{x} e^{at} b(t)\,dt + ce^{-ax}$$

is a solution of this equation. Also, show that every solution has this form.

b) Consider the equation $y'' + y' - 6y = 0$. Compute the solution ϕ satisfying $\phi(0) = 1,\ \phi'(0) = 0$. **10**

OR

c) Prove that if ϕ is any solution of **10**

$$L(y) = y'' + a_1 y' + a_2 y = 0$$

on an interval I containing a point x_0, then for all x in I

$$\| \phi(x_0) \| e^{-k|x-x_0|} \le \| \phi(x) \| \le \| \phi(x_0) \| e^{k|x-x_0|}$$

where

$$\| \phi(x) \| = \left[|\phi(x)|^2 + |\phi'(x)|^2 \right]^{1/2}, \ k = 1 + |a_1| + |a_2|.$$

d) Compute the solution ϕ of the equation **10**

$$y^{(4)} + 16y = 0$$

which satisfies

$$\phi(0) = 1,\ \phi'(0) = 0,\ \phi''(0) = 0,\ \phi'''(0) = 0.$$

UNIT – II

2. a) Let x_0 be in I, and let $\alpha_1, \ldots\ldots, \alpha_n$ be any n constants. Then prove that there is atmost one solution ϕ of **10**

$$L(y) = y^{(n)} + a_1(x)\, y^{(n-1)} + \ldots.. + a_n(x)\, y = 0 \text{ on I satisfying}$$

$$\phi(x_0) = \alpha_1,\ \phi'(x_0) = \alpha_2, \ldots\ldots, \phi^{(n-1)}(x_0) = \alpha_n$$

b) Consider the equation **10**

$L(y) = y'' + a_1(x)y' + a_2(x)y = 0$

where a_1, a_2 are continuous on some interval I. Let ϕ_1, ϕ_2 and ψ_1, ψ_2 be two bases for the solutions of L(y) = 0. Show that there is a non-zero constant k such that

$W(\psi_1, \psi_2)(x) = k\ W(\phi_1, \phi_2)(x)$.

OR

c) Find all solutions of the equation **10**

$y'' - \frac{2}{x^2}y = x, \ (0 < x < \infty)$

d) Consider the second order Euler equation **10**

$x^2 y'' + axy' + by = 0$ (a, b constants), and the polynomial q given by

$q(r) = r(r-1) + ar + b$

Prove that a basis for the solutions of the Euler equation on any interval not containing x = 0 is given by

$\phi_1(x) = |x|^{r_1}, \ \phi_2(x) = |x|^{r_2}$,

in case r_1, r_2 are distinct roots of q, and by $\phi_1(x) = |x|^{r_1}, \phi_2(x) = |x|^{r_1} \log|x|$,

if r_1 is a root of q of multiplicity two.

UNIT – III

3. a) Let g, h be continuous real-valued functions for $a \le x \le b, c \le y \le d$ respectively, and consider the equation **10**

$h(y)y' = g(x)$.

If G, H are any functions such that $G' = g, H' = h,$ and C is any constant such that the relation

$H(y) = G(x) + C$

defines a real-valued differentiable function ϕ for x in some interval I contained in $a \le x \le b$, then prove that ϕ will be a solution of

$h(y)y' = g(x)$

on I. Conversely, prove that if ϕ is a solution of

$h(y)y' = g(x)$

on I, it satisfies the relation

$H(y) = G(x) + C$

on I, for some constant C.

b) Let M, N be two real-valued functions which have continuous first partial derivatives on some rectangle **10**

$R : |x - x_0| \le a, \ |y - y_0| \le b$

then prove that the equation

$M(x, y) + N(x, y)y' = 0$

is exact in R if , and only if,

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

in R.

OR

c) Prove that if S is either a rectangle **10**

$|x-x_0| \le a, |y-y_0| \le b, (a, b > 0)$, or a strip

$|x-x_0| \le a, |y| < \infty \ (a > 0)$,

and that f is a real-valued function defined on S such that $\frac{\partial f}{\partial y}$ exists, is continuous on S, and

$\left|\frac{\partial f}{\partial y}(x, y)\right| \le k, \ ((x, y) \text{in S})$, for some k > 0. Then f satisfies a Lipschitz condition on S with Lipschitz constant K.

d) Let f be a real valued continuous function on the strip **10**

$S: |x-x_0| \le a, |y| < \infty, \ (a > 0)$, and suppose that f satisfies on S a Lipschitz condition with constant k>0. Then prove that the successive approximations $\{\phi_k\}$ for the problem

$y' = f(x, y), \quad y(x_0) = y_0$,

exist on the entire interval $|x-x_0| \le a$, and converge there to a solution of the given initial value problem.

UNIT – IV

4. a) Find the solution ϕ of **10**

$y'' = 1 + (y')^2$

which satisfies $\phi(0) = 0, \phi'(0) = 0.$

b) Suppose f is a vector-valued function defined for (x, y) on a set S of the form **10**

$|x-x_0| \le a, \ |y-y_0| \le b, (a, b > 0)$, or of the form

$|x-x_0| \le a, |y| < \infty, (a > 0)$

If $\frac{\partial f}{\partial y_k}(k = 1, \ldots\ldots, n)$ exists, is continuous on S, and there is a constant k>0 such that

$\left|\frac{\partial f}{\partial y_k}(x, y)\right| \le k, \qquad (k = 1, \ldots\ldots, n)$,

for all (x, y) in S, then prove that f satisfies a Lipschitz condition on S with Lipschitz constant K.

OR

c) Show that Euclidean length satisfies the same rules as the magnitude, namely: **10**

i) $\|y\| \ge 0$ and $\|y\| = 0$ if and only if $y = 0$,

ii) $\|cy\| = |C| \cdot \|y\|$, for any complex number c,

iii) $\|y + z\| \le \|y\| + \|z\|$

d) Compute the first four successive approximations $\phi_0, \phi_1, \phi_2, \phi_3$ for the problem **10**

$y_1^1 = y_2,$

$y_2^1 = -y_1$

$y(0) = (0, 1)$

Show that $\phi_k(x) \to \phi(x) = (\sin x, \cos x)$.

5. a) Find all solutions of the following equation **5**

$y' - 2y = 1$

b) One solution of **5**

$x^2 y'' - 7xy' + 15y = 0$

for $x > 0$ is $\phi_1(x) = x^3$. Find it's second independent solution.

c) Find all real-valued solution ϕ of **5**

$y' = x^2 y$

d) Solve the equation **5**

$y'' + y' = 1.$

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