M.Sc.-I (Mathematics) (New CBCS Pattern) Semester - I

PSCMTH05(B) - Ordinary Differential Equations

P. Pages : 4 **GUG/S/23/13742**

Time : Three Hours Max. Marks : 100

Notes : 1. Solve **all five** questions.
2. All questions carry equal marks.

UNIT – I

1. a) Let a_1, a_2 be constants, and consider the equation **10**

$$L(y) = y'' + a_1 y' + a_2 y = 0$$

If r_1, r_2 are distinct roots of the characteristic polynomial P, where

$$p(r) = r^2 + a_1 r + a_2$$

Then show that the functions ϕ_1, ϕ_2 defined by

$$\phi_1(x) = e^{r_1 x},\ \phi_2(x) = e^{r_2 x}$$

are solutions of $L(y) = 0$. Also show that if r_1 is a repeated root of p, then the functions ϕ_1, ϕ_2 defined by

$$\phi_1(x) = e^{r_1 x},\ \phi_2(x) = x e^{r_1 x}$$

are solutions of $L(y) = 0$

b) Consider the equation **10**

$$y'' + a_1 y' + a_2 y = 0$$

where a_1, a_2 are real constant such that $4a_2 - a_1^2 > 0$. Let $\alpha + i\beta, \alpha - i\beta$ $(\alpha, \beta$ real) be the roots of the characteristic polynomial.

i) Show that ϕ_1, ϕ_2 defined by

$$\phi_1(x) = e^{\alpha x} \cos \beta x,\ \phi_2(x) = e^{\alpha x} \sin \beta x$$

are solutions of the equation

ii) Compute $w(\phi_1, \phi_2)$ and show that ϕ_1, ϕ_2 are linearly independent on any interval I.

OR

c) Let $r_1, \ldots\ldots, r_s$ be the distinct roots of the characteristic polynomial p, and suppose r_i has multiplicity $m_i (m_1 + m_2 + \ldots.. + m_s = n)$. The n functions. **10**

$$e^{r_1 x}, xe^{r_1 x}, \ldots\ldots\ldots, x^{m_1 - 1} e^{r_1 x};$$

$$e^{r_2 x}, xe^{r_2 x}, \ldots\ldots\ldots, x^{m_2 - 1} e^{r_2 x};$$

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$$e^{r_s x}, xe^{r_s x}, \ldots\ldots\ldots, x^{m_s - 1} e^{r_s x}$$

are solutions of $L(y) = 0$. Prove that the n solutions of $L(y) = 0$ are linearly independent on any interval I.

d) Using the annihilator method find a particular solution of the equation $y'' + 4y = \cos x$. **10**

UNIT – II

2. a) Let x_0 be in I, and let $\alpha_1, \ldots\ldots, \alpha_n$ be any n constants. Prove that there is almost one solution ϕ of $L(y) = 0$ on I satisfying **10**

$$\phi(x_0) = \alpha_1,\ \phi^1(x_0) = \alpha_2, \ldots\ldots, \phi^{(n-1)}(x_0) = \alpha_n$$

b) Find two linearly independent solutions of the equation **10**

$$(3x-1)^2 y'' + (9x-3)\, y' - 9y = 0$$

For $x > \frac{1}{3}$

OR

c) Let b be continuous on an interval I, and let $\phi_1, \ldots\ldots, \phi_n$ be a basis for the solutions of $L(y) = 0$ on I. Prove that every solution ψ of $L(y) = b(x)$ can be written as **10**

$$\psi = \psi_p + C_1\phi_1 + \ldots\ldots + C_n\phi_n$$

where ψ_p is a particular solution of $L(y) = b(x)$ and $C_1, \ldots\ldots, C_n$ are constants. Also, show that every such ψ is a solution of $L(y) = b(x)$, where the particular solution ψ_p is given by

$$\psi_p(x) = \sum_{k=1}^{n} \phi_k(x) \int_{x_0}^{x} \frac{w_k(t)\, b(t)}{w(\phi_1, \ldots, \phi_n)(t)}\, dt$$

d) Find all solution of the equation **10**

$$x^2 y'' + 2xy' - 6y = 0 \text{ for } x > 0.$$

UNIT – III

3. a) Suppose the equation $M(x,y) + N(x,y)\, y' = 0$ is exact in a rectangle R, and F is a real – valued function such that $\frac{\partial F}{\partial x} = M, \frac{\partial F}{\partial y} = N$ in R. Prove that every differentiable function ϕ defined implicitly by a relation $F(x,y) = C$, (C = constant), is a solution of the equation $M(x,y) + N(x,y)\, y' = 0$, and every solution of this equation whose graph lies in R arises this way. **10**

b) **10**

i) Find the solution of $y' = 2y^{1/2}$ passing through the point (x_0, y_0) where $y_0 > 0$.

ii) Find all solutions of this equation passing through $(x_0, 0)$.

OR

c) Suppose S is either a rectangle **10**

$$|x - x_0| \le a,\ |y - y_0| \le b,\ (a, b) > 0)$$

or a strip

$$|x - x_0| \le a,\ |y| < \infty,\ (a > 0),$$

and that f is a real – valued function defined on S such that $\partial f / \partial y$ exists, is continuous on S, and $\left|\frac{\partial f}{\partial y}(x, y)\right| \le k,\ ((x, y) \text{ in } s),$ for some $k > 0$. Then prove that f satisfies a Lipschitz condition on S with Lipschitz constant k.

d) Consider the problem **10**

$$y' = y + \lambda x^2 \sin y,\ y(0) = 1,$$

where λ is some real parameter, $|\lambda| \le 1$.

i) Show that the solution ψ of this problem exists for $|x| \le 1$

ii) Prove that

$$\left|\psi(x) - e^x\right| \le |\lambda| (e^{|x|} - 1)$$

For $|x| \le 1$.

UNIT – IV

4. a) Solve the equation $y'' = f(y, y'),$ where f is a function independent of x. **10**

b) For any two vectors $y = (y_1, y_2, ..., y_n)$ and $z = (z_1, ..., z_n)$ in C_n define the inner product y – z to be the number given by **10**

$$y \cdot z = y_1 \bar{z} + ... + y_n \bar{z}_n$$

i) Show that $z \cdot y = (\overline{y \cdot z})$

ii) Show that $(y_1 + y_2) \cdot z = (y_1 \cdot z) + (y_2 \cdot z)$

iii) Show that if C is a complex number

$$(cy) \cdot z = c(y \cdot z) = y \cdot (\bar{c} \cdot z)$$

iv) Show that $\|y\|^2 = y \cdot y$

OR

c) Consider the system **10**

$$y_1' = 3y_1 + xy_3,$$

$$y_2' = y_2 + x^3 y_3,$$

$$y_3' = 2xy_1 - y_2 + e^x y_3$$

Show that every initial value problem for this system has a unique solution which exists for all real x.

d) Let $a_1, ..., a_n$ b be continuous complex – valued functions on an interval I containing a point x_0. If $\alpha_1, ..., \alpha_n$ are any n constants, then prove that there exists one and only one solution ϕ of the equation **10**

$$y^{(n)} + a_1(x) y^{(n-1)} + ... + a_n(x) y = b(x)$$

On I satisfying $\phi(x_0) = \alpha_1, \phi'(x_0) = \alpha_2, ..., \phi^{(n-1)}(x_0) = \alpha_n$

5. a) Find all solutions of the equation **5**

$y' - 2y = 1$

b) Define: **5**

i) Homogenous linear differential equation of order n.

ii) Non-homogeneous linear differential equation of order n.

c) Find all real – valued solutions of the equation $y' = x^2 y$ **5**

d) Solve the equation $y'' + y' = 1$. **5**

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