M.Sc. - I (Mathematics) (New CBCS Pattern) Semester - I

PSCMTH04 - Linear Algebra

P. Pages : 2 **GUG/S/23/13740**

Time : Three Hours Max. Marks : 100

Notes : 1. Solve all **five** questions.
2. All questions carry equal marks.

UNIT - I

1. a) Let V be a vector space and W a subset of V. Then prove that W is a subspace of V if and only if the following three conditions hold for the operations defined in V. **10**
i) $0 \in W$
ii) $x + y \in W$ whenever $x \in W$ and $y \in W$
iii) $cx \in W$ whenever $c \in F$ and $x \in W$

b) Let S be a linearly independent subset of a vector space V, and let ν be a vector in V that Is not is S. Then prove that $SU\{\nu\}$ is linearly dependent if and only if $\nu \in \text{span}(S)$. **10**

OR

c) Let V be a vector space and $u_1, u_2, ..., u_n$ be distinct vectors in V. Prove that $\beta = \{u_1, u_2, ..., u_n\}$ is a basis for V if and only if each $\nu \in V$ can be uniquely expressed as a linear combination of vectors of β. **10**

d) State and prove Replacement theorem. **10**

UNIT - II

2. a) State and prove dimension theorem. **10**

b) Let V and W be vector spaces over F, and suppose that $\{v_1, v_2, ..., v_n\}$ is a basis for V prove that for $W_1, W_2, ..., W_n$ in W, there exists exactly one linear transformation $T : V \to W$ such that $T(v_i) = w_i$ for $i = 1, 2, ..., n$. **10**

OR

c) Let V and W be finite-dimensional vector spaces (over the same field). Then prove that V is isomorphic to W if and only if $\dim(V) = \dim(W)$. **10**

d) Prove that the solution space for $y' + a_0 y = 0$ is of dimension 1 and has $\left\{e^{-a_0 t}\right\}$ as a basis. **10**

UNIT - III

3. a) Let T be a linear operator on a finite-dimensional vector space V, and let λ be an eigen value of T having multiplicity m. Then prove that $1 \le \dim(E_\lambda) \le m$. **10**

b) Find all the eigen vectors of the matrix. **10**

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$$

OR

c) Prove that a linear operator T on a finite dimensional vector space V is diagonalizable if and only if V is the direct sum of the eigen spaces of T. **10**

d) State and prove Cayley-Hamilton theorem. **10**

UNIT - IV

4. a) Let V be an inner product space over F. Then prove that for all $x, y \in V$ the following statements are true. **10**

i) $|\langle x, y\rangle| \le \|x\| - \|y\|$

ii) $\|x + y\| \le \|x\| + \|y\|$

b) Let V be a finite-dimensional inner product space over F, and let $g : V \to F$ be a linear transformation. Prove that there exists a unique vector $y \in V$ such that $g(x) = \langle x, y\rangle$ for all $x \in V$. **10**

OR

c) Let T be a linear operator on a finite dimensional vector space V such that the characteristic polynomial of T splits. Suppose that λ is an eigen value of T with multiplicity m. Then prove that. **10**

i) $\dim(k_\lambda) \le m$

ii) $k_\lambda = N\left((T - \lambda I)^m\right)$

d) Let p(t) be a minimal polynomial of a linear operator T on a finite-dimensional vector space V. Prove that **10**

i) For any polynomial g (t), if g (t) = T_0, then p(t) divides g(t). In particular, p(t) divides the characteristic polynomial of T.

ii) The minimal polynomial of T is unique.

5. a) Prove that: if x, y and z are vectors in a vector space V such that x + z = y + z, then x = y. **5**

b) Sow that the linear transformation $T : F^2 \to F^2$ defined by $T(a_1, a_2) = (a_1 + a_2, a_1)$ is one-to-one and onto. **5**

c) Let $A \in M_{n\times n}(F)$. Prove that a scalar λ is an eigen value of A if and only if $\det(A - \lambda I_n) = 0$. **5**

d) Let $A \in M_{m\times n}(F)$. Then prove that $\text{rank}(A^* A) = \text{rank}(A)$. **5**

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