M.Sc. I (Mathematics) (NEW CBCS Pattern) Sem-I

PSCMTH04: Linear Algebra

P. Pages : 2 **GUG/W/22/13740**

Time : Three Hours Max. Marks : 100

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Notes : 1. Solve all five questions.
2. All questions carry equal marks.

UNIT – I

1. a) Prove that, in any vector space V, the following statements are true: **10**
i) $ox = 0$ for each $x \in V$
ii) $(-a)x = -(ax) = a(-x)$ for each $a \in F$ and each $x \in V$
iii) ao = 0 for each $a \in F$

b) Prove that the span of any subset S of a vector space V is a subspace of V that contain S. Moreover, also prove that, any subspace of V that contains S must also contain the span of S. **10**

OR

c) State and prove replacement theorem. **10**

d) Prove that : If W is a subspace of a finite – dimensional vector space V, then W is finite – dimensional and $\dim(W) \leq \dim(V)$. Moreover if, $\dim(W) = \dim(V)$, then $V = W$. **10**

UNIT = II

2. a) Let V and W be finite – dimensional vector spaces of equal dimension, and let $T: V \to W$ be linear. Then prove that the following are equivalent. **10**
i) T is one – to – one
ii) T is onto
iii) rank (T) = din (V)

b) Let T be an invertible linear transformation from V to W. Then prove that V is finite – dimensional if and only if W is finite – dimensional. In this case, dim (V) = din (W) **10**

OR

c) Let V and W be finite – dimensional vector spaces over F of dimensions n and m, respectively, and let β and γ be ordered bases for V and W, respectively. Then prove that the function $\phi_\beta^\gamma : L(V, W) \to M_{m\times n}(F)$, defined by $\phi_\beta^\gamma(T) = [T]_\beta^\gamma$ for $T \in L(V, W)$, is an isomorphism. **10**

d) Prove that the differential operator $D - CI : C^\infty \to C^\infty$ is onto for any complex number C. **10**

UNIT – III

3. a) Find all eigenvectors of the matrix. **10**

$$A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$$

b) Prove that the characteristic polynomial of any diagonalizable linear operator on a vector space V over a filed F splits over F. **10**

OR

c) Show that $A = \begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}$ is diagonalizable and find a 2×2 matrix θ such that $\theta^{-1} A \theta$ is a diagonal matrix. Use this result to compute A^n for any positive integer n. **10**

d) State and prove Cayley – Hamilton theorem. **10**

UNIT – IV

4. a) Let V be a finite – dimensional inner product space, and let T be a linear operator on V. Then prove that there exists a unique function $T^* : V \to V$ such that $\langle T(x), y \rangle = \langle x, T^*(y) \rangle$ for all $x, y \in V$. Furthermore, show that T^* is linear. **10**

b) Let T be a linear operator on a finite – dimensional inner product space V. If T has an eigenvector, then prove that so does T^*. **10**

OR

c) Let T be a linear operator on a finite – dimensional vector space V such that the characteristic polynomial of T splits. Suppose that λ is an eigenvalue of T with multiplicity m. Then prove that. **10**

i) $\dim(k_\lambda) \le m$

ii) $k_\lambda = N\left((T - \lambda I)^m\right)$

d) Let T be a linear operator on a finite – dimensional vector space, V and let $P(t)$ be the minimal polynomial of T. Prove that a scalar λ is an eigenvalue of T if and only if $p(\lambda) = 0$ **10**

5. a) Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S_1 is linearly dependent, then prove that S_2 is linearly dependent. **5**

b) Let V and W be vector spaces, and let $T : V \to W$ be linear. Then prove that T is one – to – one if and only if $N(T) = \{0\}$. **5**

c) Find the eigenvalues of $A = \begin{pmatrix} i & 1 \\ 2 & -i \end{pmatrix}$ for F = C. **5**

d) Prove that in an inner product space V $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$ for all $x, y \in V$. **5**

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