M.Sc. (Mathematics) (NEW CBCS Pattern) Sem-I

PSCMTH02 - Paper-II : Real Analysis-I

P. Pages : 2 **GUG/W/22/13738**

Time : Three Hours * 3 4 4 2 * Max. Marks : 100

Notes : 1. Solve all **five** questions.
2. All questions carry equal marks.

UNIT – I

1. a) Prove that the sequence of functions $\{f_n\}$, defined on E, converges uniformly on E iff for every $\in > 0$ there exists an integer N such that $m \geq N,\ n \geq N, x \in E$ implies $|f_n(x) - f_m(x)| \leq \in$ **10**

b) Suppose $\{f_n\}$ is a sequence of functions, differentiable on [a, b] and such that $\{f_n(x_0)\}$ converges for some point x_0 on [a,b]. If $\{f_n'\}$ converges uniformly on [a,b] then prove that $\{f_n\}$ converges uniformly on [a,b], to a function f, and $f'(x) = \lim_{n\to\infty} f_n'(x)\ (a \leq x \leq b)$. **10**

OR

c) Prove that there exists a real continuous function on the real line which is nowhere differentiable. **10**

d) If K is compact, if $f_n \in \tau(k)$ for n = 1, 2, 3…… and if $\{f_n\}$ is point wise bounded and equicontinuous on K, then prove that **10**

i) $\{f_n\}$ is uniformly bounded on K.

ii) $\{f_n\}$ contains a uniformly convergent subsequence.

UNIT – II

2. a) Suppose f maps an open set $E \subset R^n$ into R^m and f is differentiable at a point $x \in E$. Then prove that the partial derivative $(D_j f_i)$ (x) exist and $f'(x)e_j = \sum_{i=1}^{m} (D_j f_i)(x) u_i,\ 1 \leq j \leq n$ **10**

b) If f is a differentiable mapping of a connected open set $E \subset R^n$ into R^m, and if $f'(x) = 0$ for every $x \in E$. prove that f is constant in E. **10**

OR

c) Suppose f maps an open set $E \subset R^n$ into R^m, then prove that $f \in \tau^1(E)$ if and only if the partial derivatives $D_j f_i$ exist and are continuous on E for $1 \leq i \leq m,\ 1 \leq j \leq n$. **10**

d) Suppose f is a τ^1- mapping of an open set $E \subset R^n$ into R^n, $f'(a)$ is invertible for some $a \in E$ & b = f(a) then prove that there exist open sets U & V in R^n such that $a \in U,\ b \in V$, f is one-to-one on U & f(U) = V **10**

UNIT – III

3. a) Let $\{U_\alpha, \phi_\alpha\}$ be an atlas on a locally Euclidean space. If two charts (V, φ) & (W, σ) are both compatible with the atlas $\{(U_\alpha, \phi_\alpha)\}$ then prove that they are compatible with each other. **10**

b) Prove that the following are Smooth manifolds **10**

i) Euclidean space
ii) Open subset of a manifold
iii) Manifolds of dimension zero
iv) Graph of a smooth function

OR

c) Prove that any atlas $\mu = \{(U_\alpha, \phi_\alpha)\}$ on a locally Euclidean space is contained in a unique maximal atlas. **10**

d) Prove that, the real line with two origins and sphere with hair are not topological manifolds. **10**

UNIT – IV

4. a) Prove that **10**

i) Suppose $F: N \to M$ is C^∞ at $p \in N$. If (U, ϕ) is any chart about p in N & (U, φ) is any chart about f(p) in M, then $\Psi o F o \phi^{-1}$ is C^∞ at (ϕ, p).

ii) If $F: N \to M$ & $G: M \to P$ are C^∞ maps of manifolds, then the composite $GoF: N \to P$ is C^∞.

b) Let N & M be smooth manifolds, and $F: N \to M$ a continuous map. Then prove that following are equivalent **10**

i) The map $F: N \to M$ is C^∞.

ii) There are atlases μ for N & μ for M such that for every chart (U, ϕ) in μ and (V, Ψ) in μ, the map $\Psi o F o \phi^{-1} : \phi\left(U \cap F^{-1}(U)\right) \to R^m$ is C^∞

iii) For every chart (U, ϕ) on N and (V, Ψ) on M, the map $\Psi \circ F \circ \phi^{-1} : \phi\left(U \cap F^{-1}(U)\right) \to R^m$ is C^∞

OR

c) Define diffeomorphism of manifolds, and further prove that **10**

i) If (U, ϕ) is a chart on a manifold M of dimension n, then the coordinate map $\phi: U \to \phi(U) \tau \mathbb{R}^n$ is a diffeomorphism

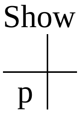
ii) Let U be an open subset of a manifold M of dimension n. If $F: U \to F(U) \tau \mathbb{R}^n$ is a diffeomorphism onto an open subset of $\mathbb{R}^n$, then (U, F) is a chart in the differentiable structure of M.

d) State & prove inverse function theorem for manifolds **10**

5. a) If $\{f_n\}$ be a sequence of continuous function on E and $f_n \to f$ uniformly on E Then prove that f is continuous on E **5**

b) State inverse function theorem **5**

c) Show that the cross **5**



with the subspace topology is not locally Euclidean at the intersection p & so cannot be a topological manifold.

d) Suppose $\left(U, x^1, x^2 --- x^n\right)$ is a chart on a manifold. Then prove that $\frac{\partial x^i}{\partial x^j} = \delta^i_j$ **5**

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