M.Sc. (Mathematics) (CBCS Pattern) Sem-I

PSCMTHT04 - Paper-IV : Linear Algebra & Differential Equations

P. Pages : 3 **GUG/W/22/11169**

Time : Three Hours Max. Marks : 100

Notes : 1. Solve all **five** questions.
2. Each question carries equal marks.

UNIT – I

1. a) Let A be an operator then prove the following statements are equivalent. **10**

i) $\det A \neq 0$

ii) ker A = (0)

iii) A is one-one

iv) A is onto

v) A is invertible

b) Let $\{e_1, e_2, ..., e_m\}$ be the basis of a vector space. If $\{v_1, v_2, ..., v_r\}$ is a linearly independent set in E then prove that $r \leq m$. **10**

OR

c) Let T be an operator for an n-dimensional vector space E. If the characteristic polynomial of T – has n distinct real roots. Then prove that, T can be diagonalized. **10**

d) Solve the initial value problem for the system **10**

$x_1' = x_1, \; x_2' = x_1 + 2x_2, \; x_3' = x_1 - x_3$

$x(0) = (u_1, u_2, u_3)$

UNIT – II

2. a) Let $T : E \to E$ be an operator on a non real vector space with distinct non real eigen values $\mu_1, \bar{\mu}_1, \mu_2, \bar{\mu}_2, ... \mu_s, \bar{\mu}_s$ then prove that there is an invariant direct sum decomposition for E and a corresponding direct sum decomposition for T. **10**

$E = E_1 \oplus E_2 \oplus ------ \oplus E_s$

$T = T_1 \oplus T_2 \oplus ------- \oplus T_s$

Such that each E_i is 2-Dimensional and $T_i \in L(E_i)$ has eigen value $\mu_i, \bar{\mu}_i$

b) Let P, S, T denotes operator on R^n then prove that, **10**

i) If $Q = PTP^{-1}$ then $e^Q = Pe^T P^{-1}$

ii) If ST = TS then $e^{S+T} = e^S \cdot e^T$

iii) $e^{-5} = (e^s)^{-1}$

OR

c) Let $\sum_{j=0}^{\infty} A_j = A$ and $\sum_{k=0}^{\infty} B_k = B$ be an absolutely convergent series of operator on R^n then prove that $A \cdot B = C = \sum_{\ell=0}^{\infty} C\ell$ Where, $C_\ell = \sum_{j+k=\ell} A_j B_k$ **10**

d) Let $T_i R^n \to R^m$ be a linear transformation then show that, t is uniformly continuous. **10**

UNIT – III

3. a) For operator T, find basis for the generalized eigen spaces, given the matrix of the semisimple nilpotent parts of T. where, $T_0 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. **10**

b) Verify each of the following operator is nilpotent and find its canonical forms. **10**

i) $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$ ii) $\begin{bmatrix} 0 & 2 & -2 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

OR

c) Let λ_y, λ_z be the roots of polynomial $\lambda^2 + a\lambda + b$ then prove that every solution of the differential equation $s'' + a_s{}' + b_s = 0$ is of the following types- **10**

Case – I : λ_1, λ_2 are real and distinct then $s(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$

Case – II : $\lambda_1 = \lambda_2 = \lambda$ real then $s(t) = c_1 e^{\lambda t} + c_2 t e^{\lambda t}$

Case – III : $\lambda_1 = \bar{\lambda}_2 = u + iv, v \neq 0 : s(t) = e^{ut}(c_1 \cos vt + c_2 \sin v t)$

d) Let $T \in L(E)$ where E is complex if T has non-real eigen value. Then prove that, t = S + N where SN = NS, S is diagonalizable and N is nilpotent. **10**

UNIT – IV

4. a) If every solution of $x' = Ax \to 0$ as $t \to \infty$ then prove that every eigen value of A has negative real part. And let x(t) be any solution of $x' = Ax$ then prove that $\lim_{t \to 0} x(t) = 0$ **10**

b) Let $A \in L(R^n)$ and let x(t) be a solution of $x' = Ax$ then prove that each coordinate $x_j(t)$ of the solution is linear combination of the function $t^k e^{ta} \cos bt$, $t^l e^{ta}$ sinbt. Where a + ib runs through all the eigen value of A with $b \geq 0$ and k and l run through all the integer 0 to n-1 moreover for each d = a+ib k and l are less than the size of the longest d-block in the real canonical form of A. **10**

OR

c) The set $\delta_2 = \{T \in L(R^n) / e^{tT}$ is hyperbolic flow$\}$ is open and dense in $L(R^n)$. **10**

d) If $A \in L(E)$ then prove the following are equivalent. **10**

a) The origin is a source for the dynamical system $x' = Ax$

b) For any norm on E, there are constants L > 0, a > 0 such that

$|e^{tA} x| \geq Le^{ta} |x|, \forall t \geq \cdot 0, x \in E$

c) There exist a > 0 and a basis $\beta(E)$ whose corresponding norm satisfies

$|e^{tA} \cdot x|\beta \geq e^{ta} |x|\beta, \forall t \geq 0, x \in E$

5. a) Let A be an n x n matrix having n distinct real eigen values $\lambda_1, \lambda_2 ......, \lambda_n$. Then prove that there exists an invertible n x n matrix Q, such that $QAQ^{-1} = \text{diag}\ \{\lambda_1, \lambda_2, ..., \lambda_n\}$ **5**

b) Let (E, N) be a normed vector space then prove that the unit ball $D = \{x \in E$ such that $N(x) \leq 1\}$ is compact. **5**

c) Let A be any operator on real or complex vector space let it's characteristic polynomial be $p(t) = \sum_{k=0}^{n} a_k t^k$ then prove $p(A) = 0$ i.e. $\sum_{k=0}^{n} a_k A^k = 0\ \forall t \in E$ **5**

d) Find Jordan / real form for the operator ton real vector space $\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ **5**

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